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MATHEMATICS FOR CANADIANS

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MATHEMATICS FOR CANADIANS

Book III

BY

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A CHAT WITH THE STUDENT

You are now about to make a second approach to algebra. It is assumed that you have acquired from your previous work certain facts and skills. If these facts have become hazy in your mind, you will soon have an opportunity to recall them; if the skills have become rusty, you will be given practice in reburnishing them. For the present, it is well for you to cast a quick glance towards the past and then set your sights on future objectives. Let us consider briefly some notions that may give you a background for the present course.

Some Uses of Algebra

In a previous book,[1] we considered at some length the question *Why Study Algebra?* It is to be hoped that you have satisfied yourself that the fruits of studying algebra are worth the effort required of you. You can hardly have failed to notice that science and technology are foundation stones in our social structure. Those who care for our many machines need a working knowledge of mathematics, while those who design these machines require a deeper knowledge of the subject. Newspapers, magazines, and catalogues make constant use of graphs, tables of values, and formulas in dealing with financial news, income tax problems, imports and exports, the cost of living, and countless other matters. Again and again you are reminded that we live in the Age of Science. Many of the sciences are founded on mathematics and their laws are expressed in the language of algebra. A dramatic example is Einstein's formula for the release of atomic energy, $E = mc^2$.

In the light of these facts, it seems reasonable that, for all those with sufficient ability, some acquaintance with algebra can be justified as one of the requirements for competent citizenship. You can readily recall some of the benefits that may result from the study of algebra. For example:

[1] *Mathematics for Canadians*, Book I.

(1) *Algebra helps to clarify the ideas of arithmetic.* It is one thing to manipulate numbers blindly but quite another thing to know the *why* as well as the *how*. A study of algebra may provide insight into the meanings of arithmetic and thus enable you to use it more intelligently.

(2) *Algebra offers ways of dealing with related quantities.* Life presents countless examples of quantities one of which depends on others: the postage depends on the weight of the parcel, the amount of gasoline consumed depends on the average speed, the amount of fruit in an orange depends on the thickness of its skin and so on. The precise nature of such a relation is often shown by the formulas and graphs of algebra. For example, the formula $V = \frac{4}{3}\pi r^3$ tells you the way in which the volume of a sphere depends on its radius.

(3) *Algebra helps to solve certain kinds of problems.* Frequently the solution of a problem involves the finding of an unknown number. The algebraic equation may then provide a method of obtaining the answer.

(4) *Algebra offers opportunities for practice in clear thinking.* The ideas and skills of algebra are precise and its methods require that correct conclusions be drawn from given data and accepted principles. Algebra presents a constant challenge to the student to think clearly and make correct judgments.

All of the foregoing reasons for the study of algebra apply to the present course with even greater force than to an introductory course. Moreover, you will doubtless be able to add other reasons; for example, algebra is useful or essential in your other school subjects such as geometry, physics, and chemistry; a sound foundation in algebra is needed for more advanced study in mathematics, as well as for engineering, navigation, and many other sciences and professions.

Aims of the Present Course

The time has come for you to take a somewhat broader view of the study of algebra; for those with adequate ability, algebra

is worth study for its own sake. The symbols and logic of mathematics exemplify "a stern perfection such as only the greatest art can show."[2] The present course is primarily intended for students who may wish to take it for one or both of the following reasons:

(1) because they enjoy coming to grips with the ideas and skills of the subject, or

(2) because they require a mastery of algebra for more advanced work in some field of knowledge.

Part of the work of the course is *practical;* it may tie in with problems of immediate usefulness or interest. Examples are the clarification of certain ideas in arithmetic, applications to the figures of geometry and the problems of physics, the concepts of ratio and proportion, the interpretation of graphs, and so on. For those proceeding to more advanced work in mathematics, engineering, the sciences, statistics, etc., the exercises on the transformation of expressions, and those dealing with fractions, radicals, quadratics, and systems of equations are very practical indeed. Facility in such operations is a necessary foundation for their future work.

However, at certain points we shall find it necessary to introduce for purposes of practice artificial problems which may take the form of interesting puzzles. Besides providing practice, these problems will often be good fun: many people enjoy puzzles. You will be wise to attack all problems of the course with determination, whether they appear to be practical or not. The fact is that you have now reached the point in your study of algebra where very little progress can be made if you insist upon the justification of each new topic on a basis of immediate usefulness. *Algebra is a system of ideas* and as such must be developed systematically. This means that oftentimes foundations must be laid for future progress, and the operations and ideas accepted as part of a pattern later to be revealed.

[2]Bertrand Russell.

In this course, you will be given an opportunity to review and extend the information contained in your first course. In addition, new concepts and methods will be introduced with a view to laying sound foundations for further study of mathematics and other related subjects. *Throughout the book, ideas will be emphasized.* You will, of course, need to acquire certain manipulative skill by means of drill exercises. Nothing could be more futile, however, than to restrict your study to the manipulation of meaningless symbols. The manipulation of letters is a means to an end; the end is concerned with an order of ideas.

Target

1. ***The understanding of certain ideas.***
   Algebra is a way of thinking.
2. ***The mastery of certain skills.***
   Know ***how*** *as well as* ***why.***

What Skills and Ideas are Assumed?

In your first contact with algebra, you encountered a number of basic notions and methods. You should know:

(1) how to add, subtract, multiply and divide literal numbers
(2) how to add, subtract, multiply and divide signed numbers
(3) how to substitute;
(4) how to solve certain types of equations;
(5) how to solve certain types of problems;
(6) how to use formulas;
(7) how to draw and interpret graphs.

These ideas are all very important and you will make little progress until they are mastered. For this reason, the next chapter gives you an opportunity to review the above-mentioned material before proceeding with new work. Be sure to take full advantage of this opportunity but before doing so, read the following carefully.

A Word of Advice

Your powers as a student will develop as you gain independence. As you learn to depend on yourself, you will grow in mental stature. In the pages that follow, you may encounter some difficulties. These can be solved either by first tackling them aggressively yourself or by going at once to the teacher for help. The first method is the method of mastery.

Never ask for more help than you need.

Always try first to solve a problem by yourself. If you fail, *read the text carefully until you grasp clearly the ideas involved.* You will find many examples worked in the text. Try them yourself, looking at the given solutions only when absolutely necessary. The next step is to attack the exercises which follow, and show your finished effort to your instructor. If you have done your best and failed, then get help.

You will find it a good idea to get into the habit of checking your work. PRACTISE MAKING ESTIMATES TO DETECT GROSS ERRORS. Often the work may be checked by substituting numbers for the letters; substitution verifies the roots of an equation. Patience and constant checking will help you become an accurate worker.

Once you gain independence so that you can win most of your battles without outside help, you will find a new satisfaction in your work. In addition, you will develop the mastery and self-confidence that are indispensable for future study.

ALGEBRA

CHAPTER I

REVIEW OF BASIC SKILLS

Substitution, Addition, and Subtraction

The letters of algebra stand for numbers. Thus, in a given expression, if we know the numbers for which the letters stand, we can find the value of the expression by ***substitution.***

Example. If $x = -1$, $y = -2$, and $z = 3$, find the value of (*a*) $x^2 + y^2$, (*b*) $(x + y)^2$, (*c*) $x^3 + y^3 + z^3 - 3xyz$.

Substituting, we have

$$x^2 + y^2 = (-1)^2 + (-2)^2 = 1 + 4 = 5,$$

$$(x + y)^2 = (-1 - 2)^2 = (-3)^2 = 9,$$

$$x^3 + y^3 + z^3 - 3xyz = (-1)^3 + (-2)^3 + 3^3 - 3(-1)(-2)3$$
$$= -1 - 8 + 27 - 18 = 0.$$

If the literal parts of two terms are the same, the terms are called ***like;*** thus $3x^2y$, $-2x^2y$, and $\frac{3}{4}x^2y$ are like terms.

> *The sum (difference) of like terms is another like term whose coefficient is found by adding (subtracting) those of the given terms*

Example 1. Add $8mn$, $-3mn$, $-1.7mn$, and check the work when $m = 1$ and $n = 2$.

Since the sum of the coefficients is 3.3, we have

	Check: $m = 1$, $n = 2$.
$8mn$	$8 \times 2 = 16$
$-3mn$	$-3 \times 2 = -6$
$-1.7mn$	$-1.7 \times 2 = -3.4$
$3.3mn$	$3.3 \times 2 = 6.6$

Example 2. Subtract $3a^2 - 7a + 2$ from $7a^2 + 4a - 7$, and check when $a = 2$.

The work is conveniently carried out in a vertical arrangement.

	Check: $a = 2$
$7a^2 + 4a - 7$	$28 + 8 - 7 = 29$
$3a^2 - 7a + 2$	$12 - 14 + 2 = 0$
$4a^2 + 11a - 9$	$16 + 22 - 9 = 29$

NOTE: Be sure that the final 29 in the check can be obtained both vertically and horizontally from the arrangement.

Your experience with addition will convince you of the truth of the following basic laws.

LAW 1. *The sum of two numbers is the same regardless of the order in which the numbers are added.*

This is the ***commutative law of addition;*** in symbols, the law becomes

$$\boxed{a + b = b + a}$$

LAW 2. The ***associative law of addition*** is expressed in symbols:

$$\boxed{(a + b) + c = a + (b + c)}$$

Since, by Law 1, $b+c=c+b$, it follows that

$$a+(b+c)=a+(c+b),$$

and, by Law 2, $a+(c+b)=(a+c)+b$.

Each of the sums $(a+b)+c$, $a+(b+c)$, and $(a+c)+b$ can be denoted by $a+b+c$. *In adding three numbers, we may add any two of the numbers first, and then add the third number to their sum.*

Set No. 1: Substitution, Addition, Subtraction

If $x = 1$, $y = -2$, and $z = 1.4$, evaluate the following:

1. $x + y$.
2. $x - y$.
3. $(x + y)^2$.
4. $x^2 + y^2$.
5. $x^2 - y^2$.
6. $(x + y + z)^2$.
7. $x^3 - y^3$.
8. $y^3 + z^3$.
9. $(y + z)^2$.

10. $2x^2 - 5x + 3$. **11.** $2y^2 - 5y + 3$. **12.** $2z^2 - 5z + 3$.

13. $x^3 - 3x^2 + 5x - 3$. **14.** $3y^3 - y^2 - 4y + 7$. **15.** $3x^2 - yz$.

16. $\dfrac{y - z}{y + z}$. **17.** $\dfrac{x^2 + y^2 + z^2}{x + y + z}$. **18.** $\dfrac{x^4 + y^4}{x^2 + y^2}$.

19. $\left\{xyz - \sqrt{z^3(3x^2 + y^2 - 7x)}\right\}^2$. **20.** $x^3 + y^3 + z^3 - 3xyz$.

21. $\dfrac{xy + yz - zx}{x + y + z}$. **22.** $\sqrt{\dfrac{x^2 - y^2}{y - x}}$.

dd:

23.
$-11x$
$6x$

24.
$-7.3ab$
$-2.1ab$

25.
$-\frac{3}{4}m$
$0.5m$

26.
$7m^2$
$-2m^2$

27.
$-4xy$
$-9xy$

28.
$-4.5a^2b^2$
$8.1a^2b^2$

29.
$3x$
$-4x$
$-6x$
$5x$

30.
$-3.2pq$
$-4.8pq$
$7.6pq$
$-0.3pq$

31.
$\frac{3}{4}ab$
$-\frac{1}{2}ab$
$\frac{1}{5}ab$

ubtract:

32.
$-7x - 4.6$
$-5x - 5.8$

33.
$-3x^2 - 8y^2z$
$5x^2 - 4y^2z$

34.
$3a^2 - 6b^2 + 3$
$4a^2 + 2b^2 - 7$

35.
$2x^2 - 3x + 5$
$x^2 + 7x - 4$

36.
$12a^2 - 6a - 4$
$-5a^2 + 3a - 3$

37.
$-2.3z^2 + 8.2z - 7.6$
$3.2z^2 - 6.5z + 1.4$

ind the sum and the difference of each of the following:

38.
$3m^2 - 8m + 9$
$-m^2 + 5m - 12$

39.
$7a + 3b - 5$
$6a - 5b + 2$

40.
$12c - 8d + 3e$
$9c + 9d - 5e$

41. From the sum of $3x^2 - 7x + 5$ and $4x^2 + 3x + 7$, subtract the sum of $-2x^2 - 8x + 4$ and $12x^2 - 7x + 2$.

Collect like terms in the following:

42. $2a^2 - 7b^2 + 11ab - 4b^2 - 3ab - 5a^2$.

43. $12x - 17y + 8x - 20x + 20y$.

44. $mn - 5kn - 4km - 8mn + 6kn - km$.

45. $6x^2 + 8x - 15 - 5x^2 - 8x + 12$.

46. $11y - 8y + 15 - 5y - 8 - 9 - y + 2$.

47. $-r^3 - 8r^2 + 17s^3 + 10r^3 - 25s^3$.

Multiplication and Division

Multiplication, like addition, is performed according to ce tain basic principles. The following three laws emerge fro experience in multiplying numbers.

LAW 1. *The product of two numbers is the same regardless of t order in which they are taken.*

This is the ***commutative law of multiplication***; in symbol we have

$$\boxed{ab = ba}$$

LAW 2. The ***associative law of multiplication*** is expresse in symbols:

$$\boxed{(ab)c = a(bc)}$$

By Law 1, $a(bc) = a(cb)$, and, by Law 2, $a(cb) = (ac)b$.

Each of the equal products $(ab)c$, $a(bc)$, and $(ac)b$ can b denoted by abc. *In finding the product of three numbers, we fir multiply any two of the numbers and then multiply their produ by the third number.*

LAW 3. *To multiply the sum (difference) of two numbers by number, we may multiply each of the two numbers by t third number and add (subtract) the products.*

This is the ***distributive law of multiplication***; in symbols,

$$\boxed{\begin{aligned} a(b + c) &= ab + ac \\ a(b - c) &= ab - ac \end{aligned}}$$

The distributive law shows that the removal of parentheses an expression is accomplished by the operation of multiplica-on. ***Parentheses indicate multiplication.***

Recall that in multiplication and division of signed numbers, ***ke signs give positive answers, unlike signs give negative nswers.***

The following expansions are need'ed so frequently that you ıould commit them to memory.

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$(a+b)(a-b) = a^2 - b^2$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

xample. Multiply $2x^2 - 7x + 8$ by $3x - 2$ and check when $= 1$.

The vertical arrangement is convenient.

$$\begin{array}{r} 2x^2 - 7x + 8 \\ 3x - 2 \\ \hline 6x^3 - 21x^2 + 24x \\ \quad - 4x^2 + 14x - 16 \\ \hline 6x^3 - 25x^2 + 38x - 16 \end{array}$$

Check:	$x = 1$
$2 - 7 + 8$	$= 3$
$3 - 2$	$= 1$
$6 - 25 + 38 - 16$	$= 3$

Example 1. Simplify by removing brackets and collecting like erms:

$$(2x+3y)^2 - (2x+3y)(2x-3y) + (2x-3y)^2.$$

We have

$$\begin{aligned} &4x^2 + 12xy + 9y^2 - (4x^2 - 9y^2) + 4x^2 - 12xy + 9y^2 \\ &= 4x^2 + 12xy + 9y^2 - 4x^2 + 9y^2 + 4x^2 - 12xy + 9y^2 \\ &= 4x^2 + 27y^2. \end{aligned}$$

Example 2. Divide $6x^3 - 7x^2 - 50$ by $2x - 5$ and check wh
$x = 2$.

The work is conveniently arranged as follows:

$$
\begin{array}{r|l}
\multicolumn{1}{r}{3x^2 + 4x + 10} & \text{Check: } x = 2 \\
\cline{1-1}
2x - 5) \; 6x^3 - 7x^2 - 50 & \multicolumn{1}{r}{12 + 8 + 10} \\
\multicolumn{1}{r}{6x^3 - 15x^2} & 4 - 5) \; 48 - 28 - 50 \\
\cline{1-1}
\multicolumn{1}{r}{8x^2} & \\
\multicolumn{1}{r}{8x^2 - 20x} & \text{or} \\
\cline{1-1}
\multicolumn{1}{r}{20x - 50} & \\
\multicolumn{1}{r}{20x - 50} & \multicolumn{1}{r}{30} \\
\cline{1-1}
\multicolumn{1}{r}{0} & -1) \; -30
\end{array}
$$

Set No. 2: Multiplication and Division

Write at sight the following expansions:

1. $(m + n)^2$. **2.** $(a - x)^2$. **3.** $(c + d)\,(c -$
4. $(3m - 2n)^2$. **5.** $(x + y)^3$. **6.** $(x - 2y)^3$.
7. $(2k + 5m)^2$. **8.** $(m + n + r)^2$. **9.** $(7x - 3y)^2$.
10. $(7x - 3y)\,(7x + 3y)$. **11.** $(x - 2y - z)^2$.
12. $(3x^2 - 9y^2)\,(3x^2 + 9y^2)$. **13.** $(2a - b - 3c)^2$.
14. $(3c + 2d)^3$. **15.** $(4m - 3n)^3$.

Multiply:

16. $2x^2 - 7x + 5$ by $3x - 4$. Check when $x = 1$.

17. $5a^2 - 2ab + b^2$ by $3a^2 + ab - 4b^2$. Check when $a =$
and $b = 2$.

18. $3x^2 - 2xy + 5y^2$ by $2x^2 + 5xy - 4y^2$.

19. $x + y + z$ by $x^2 + y^2 + z^2 - xy - yz - zx$.

20. $3x - 4y$ by $9x^2 + 12xy + 16y^2$.

Divide:

21. $16x^3y^2 - 8x^2y^3 + 24xy^4$ by $-8xy^2$.

22. $4a^2b^2 + 6ab^3 - 2a^3b^2$ by $2ab^2$.

23. $27a^3 + 9a^2 - 3a - 10$ by $3a - 2$. Check your answ

24. $x^3 + 2x^2 + x + 2$ by $x + 2$. Check your answer.
25. $3x^3 - 5x^2 + 3x - 1$ by $x - 1$.
26. $a^4 + 2a^3 + 3a^2 + 2a + 1$ by $a^2 + a + 1$. Check when $= 2$.
27. $3x^4 - 5x^3y - 12x^2y^2 - xy^3 + 3y^4$ by $3x^2 + xy - y^2$. heck when $x = 1$, $y = 2$.
28. $m^3 - 8m - 3$ by $3 - m$.
29. $z^4 + 2z^3 + z + 2$ by $1 - z + z^2$. Check when $z = 2$.
30. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.
31. $a^3 + b^3$ by $a + b$.

mplify the following by performing the indicated multiplica- ons and collecting like terms.

32. $(x + y)^2 - (x + y)(x - y) + 2(x - y)^2$.
33. $(a + b + c)^2 - (a + b)^2 - (b - c)^2$.
34. $(a + b)^3 - (a - b)^3$.
35. $(3x - 4y)^2 - 3(2x + y)(2x - y) - (2x + 3y)^2$.
36. $-(m - 3n)^2 - 2(5m + 2n)(2m - n) + (5m + 4n)^2$.
37. $(x - 5)(2x^2 - 3x + 7) - (2x + 3)(x^2 + 3x - 12)$.
38. $(3x - 2)(2x^2 - 5x + 6) - (x + 4)(5x^2 - x - 2) +$ $+ 2)^3$.
39. $(a - 1)(2a + 3)(a + 5) - (2a - 5)(a + 3)(a - 3)$.
40. $(a + 2b - c)^2 + (2a - b)^2 - (b - 2c)^2 + (a + c)^2$.
41. $(x^2 - 5x + 8)(4x^2 + 3x + 3) - 2(2x^2 + 5)(x^2 + 6x - 5)$.
42. $3(2x - 3y)^2 - 5(2x + 3y)(2x - 3y) + 2x(4x - y)$.
43. $5(3x + 4)^2 - 3(5x - 7)^2 + 30(x + 10)(x - 10)$.
44. $3(x - 2y)^3 - 5(2x + 3y)^3 - 2(5x - 4y)(5x + 4y)(x - y)$.

Equations

The first equations you encountered were those containing a ngle unknown number raised to the first power only. These *rst-degree equations* were solved by using one or more basic rinciples. In addition to the fundamental laws given on page 2 nd page 4, we have the following four basic principles:

The equality of the two sides of an equation is preserved if we

(1) *add the same number to each side,*
(2) *subtract the same number from each side,*
(3) *multiply each side by the same number,*
(4) *divide each side by the same number, provided that the number is not* 0.

With the aid of these four principles, the unknown numb can be isolated on one side of the equation and thus evaluat

Example. Solve: $2.5x - (x - 3) = \frac{3}{2}(4 - x) - 4.5$.

We have

$$2.5x - x + 3 = 6 - \tfrac{3}{2}x - 4.5$$ (Remove parentheses— *distributive law*)

or $$1.5x + 3 = 1.5 - \tfrac{3}{2}x.$$ (Collect—*associative a commutative laws*)

Therefore $$3x + 6 = 3 - 3x,$$ (Multiply by 2)

$$6x = -3,$$ (Add $3x$; subtract 6)

and $$x = -\tfrac{1}{2}.$$ (Divide by 6)

Verification: Substitute in the given equation.

Left side $= -5/4 - (-7/2) = 9/4.$

Right side $= (3/2)(9/2) - 4.5 = 9/4.$

Systems of two linear equations with two unknowns m be solved by eliminating one of the unknowns as in the follo ing example.

Example. Solve the system: $3x - 4y = 8$ . . . (1)

$2x + 3y = -6$ . . . (2)

Multiplying (1) by 3, (2) by 4, and adding, we eliminate y. This gives

$$\begin{array}{rcl} 9x - 12y = & 24 & \\ 8x + 12y = & -24 & \\ \hline 17x = & 0, & \text{whence } x = 0. \end{array}$$

From (1), we obtain by substitution

$$0 - 4y = 8$$

whence $$y = -2.$$

It is readily seen that both of the given equations are satisfied by $x = 0$, $y = -2$.

When common or decimal fractions occur in equations, you ve to decide whether you prefer to work with the fractions as en or to eliminate them by multiplying both sides of the uation by a suitable number.

ample. Solve the system: $0.4x - 0.5y = -0.17$ . . . (1)
$0.5x + 2y = 1.1$ . . . (2)

If we multiply (1) by 100 and (2) by 10, the system can then be dealt with in the same way as the last example. Otherwise we can multiply (1) by 4 and add, thus eliminating y as follows:

$$\begin{array}{rcl} 1.6x - 2y &=& -0.68 \\ 0.5x + 2y &=& 1.1 \\ \hline 2.1x &=& 0.42 \end{array}$$

Therefore, $x = 0.42/2.1 = 42/210 = 1/5 = 0.2$.
Substitution in (2) gives

$$0.1 + 2y = 1.1$$

or $2y = 1$, whence $y = \frac{1}{2} = 0.5$.
The verification is left as an exercise.

Set No. 3: Equations

lve for the unknown number and verify your answers.

1. $2x + 3 = 7$.
2. $3x - 2 = 7$.
3. $\frac{1}{2}x + 5 = 4$.
4. $\frac{1}{3}x - 3 = 2$.
5. $11 = 5a - 4$.
6. $4m + 8 = -12$.
7. $\frac{3}{4}b = -6$.
8. $1.5x = 0.45$.
9. $ax = b$, where a and b are known.
10. $x + m = n$, where m and n are known.
11. $0.2x + 1 = 2$.
12. $\frac{2}{3}c + 3 = 3\frac{2}{3}$.

13. $6x - 6.2 = 12.4$.
14. $3m = m - 8$.
15. $4r + 5 = 6r + 3$.
16. $7a - 2 = 5a - 8$.
17. $\frac{3}{2}b + 9 = 3b - 6$.
18. $\dfrac{x - 2}{3} = 4$.
19. $4.2x + 1.6 = 5.8$.
20. $x/3 + 20 = x + 4$.
21. $-3n - 7 = 8 + 2n$.
22. $5x - 2(x - 3) = 3(4 - x) - 9$.
23. $\dfrac{2x + 1}{3} = \dfrac{2x + 4}{4}$.
24. $\frac{5}{2}s - (s - 3) = \frac{3}{2}(4 - s) - \frac{9}{2}$.
25. $0.5(7x - 5) = \frac{2}{3}(8x - 23)$.
26. $3z - 10 = \frac{1}{2}z - 15$.
27. $x/2 - 3 = x/3 + 1$.
28. $6x(x - 3) + 20 = 5 - 3x(5 - 2x)$.
29. $\dfrac{a - 2}{2} - \dfrac{a + 2}{4} = \dfrac{a - 3}{3}$.
30. $\dfrac{2(m - 7)}{3} = \dfrac{3m}{5} - 1$.
31. $0.5(2 - x) - 0.2(8x + 21) = x + 3$.
32. $h^2 - 3h - 5(h - 2) = 12 - (2 - 2h - h^2)$.
33. $\frac{1}{5}(x + 1) - \frac{2}{3}(x - 2) = 2$.
34. $1.7(m - 2) = 0.7 + 0.3(2m + 1)$.

Solve the following systems; verify your solutions.

35. $x - 3y = 1$
$2x + 3y = 29$.
36. $2x + 3y = 14$
$x + y = 5$.
37. $3m - 4n = 1$
$5m + 2n = 6$.
38. $4a + 5b = 2$
$5a + 4b = 7$.
39. $7x - 3y = 5$
$5x - 2y = 4$.
40. $3x - 5y = 0.5$
$2x - y = 1.5$.

41. $x + \frac{1}{2}y = 4$
$2x - y = 4.$

42. $0.3x + 0.2y = 0.1$
$0.2x + 0.3y = -0.1.$

43. $2.4x - 1.6y = 1.04$
$0.8x - 2.5y = 0.74.$

44. $\frac{3}{2}x + \frac{5}{6}y = \frac{1}{3}$
$x - \frac{2}{3}y = \frac{5}{6}.$

45. $12x - 3.8y = 4.4$
$3x + 2y = 7.$

46. $2x/7 + 5y/6 = 13/42$
$x/7 - 10y = -27/14.$

Formulas

The formula for the volume of a cylindrical tin can is

$$V = \pi r^2 h.$$

his formula is an equation giving the relation connecting the dius, height, and volume of ***any*** cylinder. Changes in one or th of r and h will produce changes in V in accordance with the uation $V = \pi r^2 h$. Since the values of V, r, and h may be fferent for different cylinders, we sometimes call these letters ***riables.*** The formula may then be thought of as a ***relation nong the variables*** V, r, and h. If measures of some ***rticular*** can are substituted for r and h, then the formula ves us the measure of its volume.

Note that V, r, and h are ***numbers***; they are the ***measures*** the volume, radius, and height, respectively. We substitute mbers in the formula and we get a number out of it. However, th each formula like $V = \pi r^2 h$ are associated certain systems units which must be consistent. Thus if we substitute the easures of the radius and height in inches, the formula will ve us the measure of the volume in cubic inches.

xample 1. Find, to 2 figures, the volume of a tin can with dius 7 cm. and height 100 cm. ($\pi = 22/7$).

We have

$$V = \pi r^2 h = \tfrac{22}{7} \times 49 \times 100 = 15{,}400.$$

Therefore the volume to two figures is 15,000 c.c.

Since the formula is an equation, we can use the basic principles of equations to transform it. For example,

if	$V = \pi r^2 h$	. . . (1)
then	$h = V/\pi r^2$	. . . (2)
and	$r^2 = V/\pi h$	
whence	$r = \sqrt{V/\pi h}$	. . . (3)

Such transformations are often useful in that they may prov a more convenient formula for a given problem. In (1) V sometimes called the ***subject of the formula,*** and such op ations are then referred to as ***changing the subject the formula.*** Thus, h is the subject in (2) and r is the subj in (3). We also say that (2) is ***solved for h*** and that (3 ***solved for r.***

Example 2. Solve (*a*) $K = aR + b$ for R; (*b*) $V = \frac{4}{3}\pi r^3$ fo

(*a*) We have $K - b = aR,$

whence $\dfrac{K - b}{a} = R.$

(*b*) Similarly, $3V = 4\pi r^3$

and $\dfrac{3V}{4\pi} = r^3.$

Therefore $r = \sqrt[3]{3V/4\pi}.$

When you really understand formulas, you can get us and important information by merely inspecting them. St the next two examples.

Example 3. The circumference of a circular wire ring is fou from the formula $c = 2\pi r$, where r is the measure of the rad What is the change in the circumference when the radius (*a*) doubled, (*b*) halved, (*c*) increased 4-fold, (*d*) quartered

The answers to these questions are simple when it is no that r is a factor of the right side of the formula. Thus if factor r is doubled, the right side of the formula is doubled. Hence the circumference is doubled. Simila

when r is halved, increased 4-fold, or quartered, the circumference is halved, increased 4-fold, or quartered, respectively.

ample 4. The area of the top of a circular cookie is given by $= \pi r^2$. What is the effect on the area when the radius is doubled, (*b*) halved, (*c*) tripled, (*d*) reduced to one-third its ginal value?

Since $r^2 = r \times r$, we see that r occurs twice as a factor of the right side of the formula. Therefore, if r is doubled, ***two*** factors on the right side are doubled so that the measure of the area becomes four times as great. Similarly, if r is halved, the area is reduced to 1/4 of its former value.

Now satisfy yourself that in (*c*) the area is increased 9-fold, and in (*d*) the area is reduced to 1/9th of its former value.

Set No. 4: Formulas

r each of the following formulas, find the value of the unown number: $(\pi = 22/7)$

1. $A = \pi(R^2 - r^2)$; $R = 4$, $r = 3$.
2. $F = \frac{9}{5}C + 32$; $C = 100$.
3. $S = \frac{n}{6}(n + 1)(2n + 1)$; $n = 20$.
4. $s = ut + \frac{1}{2}at^2$; $u = -48$, $t = 5$, $a = 32$.
5. $K = x^2yzh$; $x = 2$, $y = 5$, $z = 3$, $h = 10$.
6. $s = \dfrac{a(r^n - 1)}{r - 1}$; $a = 10$, $r = 2$, $n = 6$.
7. $n = 0.907\left\{\dfrac{D}{d} - 0.94\right\}^2 + 3.7$; $D = 3$, $d = \frac{1}{4}$.

Make a formula for each of the following rules, giving units propriate to your formula.

8. The number of units in the area of a rectangle equals the oduct of the measures of its length and width.

Solution: $A = lw$ is the required formula. If the length rods and the width w rods, then the area is A square rod

NOTE: You will frequently hear statements such as "area equals le times width". Such statements are concise and are in common use, they involve logical difficulties. You can't multiply *a length by a w* ***only numbers can be multiplied.*** So, when you say "length t width", it must be understood that "length" means "number of uni the length", and "width" means "number of units in the width." W this understanding, the shorter statements may be used without ma matical offence. They are so used in some of the following exercises.

9. The measure of the volume of a cardboard shoe-box equ the product of the measures of its length, width, and height

10. The volume of a conical paper cup is one-third of vertical height times the area of its circular top.

11. The area of tin in a length of stove-pipe equals its cumference times its length.

12. If a stone is released from the edge of a precipice, distance, in feet, through which it falls is 16 times the squar the time of fall measured in seconds.

13. To convert a temperature expressed in degrees Fahren to degrees Centigrade, subtract 32 from the number of Fahr heit degrees and multiply the remainder by 5/9.

14. The distance traversed equals the average rate of sp times the time.

15. Solve the formula $V = lwh$ for h.

16. Solve the formula $K = a + bx^2$ for x.

In each of the next five exercises, solve the formula for unknown letter before substituting.

17. $d = 16t^2$. Find t if $d = 400$.

18. $C = pn$. Find p if $C = 1250$ and $n = 25$.

19. The area of tin in a length of stove-pipe is given $A = 2\pi rh$. If the area is 440 sq. in. and the height is 20 i find the radius.

20. The area of a ball with radius r is given by the form $A = 4\pi r^2$. What is the radius of the largest ball that could covered with 616 sq. in. of tinfoil?

:1. The area of a trapezium with height h units and parallel
es a and b units, respectively, is given by $A = \frac{h}{2}(a + b)$.
$A = 60$, $a = 12$, $b = 8$, find h.

:2. *An aeroplane formula.* The air pressure in pounds per
ft. on the wings of an aeroplane travelling at v m.p.h. is given
proximately by $p = 0.005v^2$. What is the effect on p when
v is doubled? (*b*) v becomes one-third its original value?
v is increased to 3/2 times its first value?

:3. *An ocean liner formula.* The cost, C, of building a liner is
en approximately by $C = ks^3$, where k is a number depending
the type of ship and s is its top speed. What change is pro-
ed in the cost by (*a*) doubling the desired speed? (*b*) trebling
speed? (*c*) halving the speed?

:4. *A radio formula.* If W is the wave-length in metres and
he number of kilocycles, a formula used in radio work gives
$= 300{,}000/k$. What is the effect on W when (*a*) k is doubled?
k is trebled? (*c*) k is quartered?

Construct formulas for the perimeters and areas of the shaded
ions in the accompanying figures.

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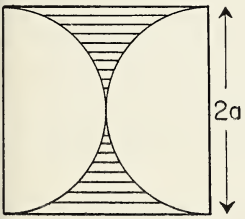

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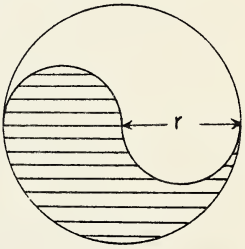

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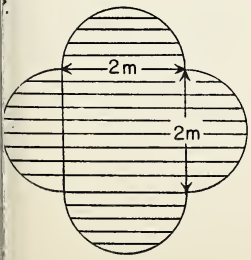

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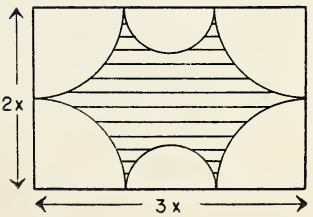


29. Use the following table to make a formula connec x and y.

y	2.5	5	7.5	10	12.5	15	17.5	20
x	1	2	3	4	5	6	7	8

30. Use the following table to find a formula showing c depends on w.

Weight of letter in oz. (w)	1	2	3	4	5	
Cost of postage in cents (c)	4	6	8	10	12	1

31. Suppose that a car with good brakes is travelling r m.p.h. on a dry pavement. Then the shortest distance in in which the car can stop is given approximately by the form

$$d = 0.045r^2 + 1.1r.$$

Find d when (a) $r = 30$, (b) $r = 60$. Why does the doubling produce more change in d than doubling?

32. The edge of a cubical cardboard carton is x inches. W a formula for (a) the area of the box when closed, (b) the volu of the box. If x is doubled, what change is produced in (c) area? (d) the volume? Provided that the cardboard doesn't g way, why is it cheaper to pack certain goods in large boxes t in small?

33. For people over 5 feet in height, the normal weight pounds, should be approximately that given by the formula

$$w = 5.5(h - 60) + 110,$$

where h is the number of inches in height. According to formula, what should be the weight of a person (a) 64 inches t (b) 72 inches tall?

Problems

If a problem involves the finding of one or two unknown
ibers, the solution may be found by setting up one or two
ations and solving them.

imple. A farmer uses a feed mixture containing shelled corn
oats. He finds that 10 bushels of this mixture weigh 494 lb.
wonders how many bushels of each cereal the mixture con-
is. If a bushel of oats weighs 34 lb. and one of shelled corn
ghs 56 lb., find the solution to the farmer's problem.

Solution 1: *With one unknown.*
Let the number of bushels of oats be x. Then the number of bushels of corn is $10 - x$. Since the oats weigh $34x$ pounds and the corn $56(10 - x)$ pounds, we have

$$34x + 56(10 - x) = 494.$$

Therefore
$$34x + 560 - 56x = 494,$$
$$-22x = -66,$$

and
$$x = 3.$$

The mixture contains 3 bushels of oats and 7 bushels of corn.

Solution 2: *With two unknowns.*
Let the mixture contain x bushels of oats and y bushels of shelled corn. Then we have

$$x + y = 10$$

and
$$34x + 56y = 494.$$

If you solve this system in the usual way, you will find that $x = 3$ and $y = 7$, which leads to the same answer as before.

In many problems, the question of whether to use one unwn or two is simply a matter of personal preference. How-r, some problems are better adapted to one method than to other. In all problems of this kind, the steps in the solution usually be taken in the following order:

(1) *Read the problem carefully to find out what is needed.* (Sometimes a diagram is helpful in picturing the facts.)

(2) *Let some letter, say x, stand for an unknown number there are two unknown numbers, you may introdu second letter for the other one.*
(3) *Using the letter-number(s), translate the English statem into algebraic equations.*
(4) *Re-read the problem and find what expressions are e*
(5) *Form an equation from the equal expressions; if two knowns are used, form two equations.*
(6) *Solve the equation or the system, as the case may be.*
(7) *Test whether the answer satisfies the conditions of problem.*

Note carefully that ***the letters stand for numbers.*** T in step (2), statements such as "Let the money be x", "Let the men", should be changed to "Let x be the number of dolla "Let x be the number of men", respectively. The literal nu often stands for the measure of a quantity; e.g. x years, x t $\$x$, x miles per hour, etc. In such cases, be sure that the pr unit is indicated.

In the following exercises, you will have an opportunit make use of equations to solve simple problems. Some of problems will require you to recall important facts f geometry.

Set No. 5: Problems

Exercises for oral answers

If x represents an unknown number, translate the follo phrases and statements into algebraic notation:

1. 17 added to an unknown number.
2. A certain number decreased by 12.
3. A certain number trebled.
4. An unknown number squared.
5. An unknown number quadrupled and the result decre by 9.
6. An unknown number increased by 5 and the re doubled.

7. A certain number squared and the result increased by 17.
8. One-third of a certain number, decreased by 4.
9. 67 added to three times a certain number.
10. If 18 is subtracted from a certain number, the result is 42.
11. If triple an unknown number is decreased by 7, the result 34.
12. A certain number is doubled when 10 is added.
13. A certain number is trebled when 12 is added.
14. If the square of a certain number is increased by 7, the sult is 43.
15. 25% of a certain number equals the number decreased 4.

Exercises for written answers

16. What number is 12 greater than x? 12 less than x? x less an 12? one-twelfth of x? 5 more than 3 times x?
17. If a number is trebled and the result increased by 18, the m is 69. Find the number.
18. If a number is quadrupled and the result decreased by 23, e remainder is 77. Find the number.
19. The sum of two numbers is 32. Twice the first number creased by three times the second is 78. Find the numbers, ing two methods.
20. The perimeter of an isosceles triangle is 28 inches. If the se is 12 inches less than the sum of the two equal sides, find e length of one of the equal sides.
21. One angle of a triangle is twice as large as the smallest gle and the third angle is 48° less than the sum of the other o. Find the angles.
22. The perimeter of a rectangular field is 120 rods. The sum the length and twice the width is 85 rods. Find the length and e width, using two methods.
23. John's father is 3 times as old as John was two years ago. ne sum of their ages today is 62 years. How old is John?
24. Find two consecutive whole numbers whose sum is 97.

25. How much money invested at 2% per annum will yi interest amounting to \$20.40 in one year?

26. When two straight lines intersect, one of the angles formed contains $x°$ and an adjacent angle contains $(4x - 20$ Find, in degrees, the four angles at the intersection.

27. George went to a post-office to buy \$10 worth of 3-c and 4-cent stamps. If he purchased a total of 300 stamps, h many of each did he buy?

28. The larger of two numbers is 9 more than 7 times t smaller, and their sum is 105. Find the numbers.

29. A box contains 140 coins having a total value of \$8.75. the coins are all either nickels or dimes, how many of each ki are there?

30. A grocer mixes tea worth 50c. a pound with another bra worth 60c. a pound to obtain a mixture worth 54c. a pou How many pounds of each brand does he use to obtain 100 pou of mixture?

31. A taxi-cab charges 20 cents for the first quarter-mile a 10 cents for every quarter-mile thereafter. How far can y ride for \$1.50?

32. A lead pipe 28 feet long is cut into two parts so that t larger part is four times as long as the smaller. How long the parts?

33. Find four consecutive integers whose sum is 166.

34. In an isosceles triangle, an angle opposite one of the eq sides is 20° greater than three times the angle opposite the ba Find the angles of the triangle.

35. A motorist travelled a distance of 205 miles in 6.5 hou Part of the trip was made on good road over which he averag 35 m.p.h., and the rest of the trip was made at an average 20 m.p.h. because the road was under construction. How ma miles of the road were under construction? (*Hint:* Find t number of hours travelled on each kind of road. Recall t formula $d = rt$.)

CHAPTER II

GRAPHS

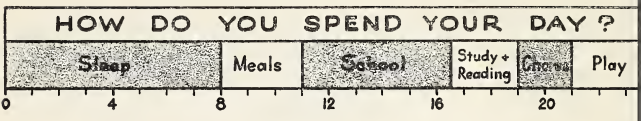
It is assumed in this course that the student has had some xperience with graphs of various kinds. The purpose of the resent chapter is to review some of the earlier ideas and to stress ertain aspects of graphs with a view to developing a deeper isight into their nature and use. To this end, we shall consider rst some notions which may help to bring the meaning of the raph into sharper focus.

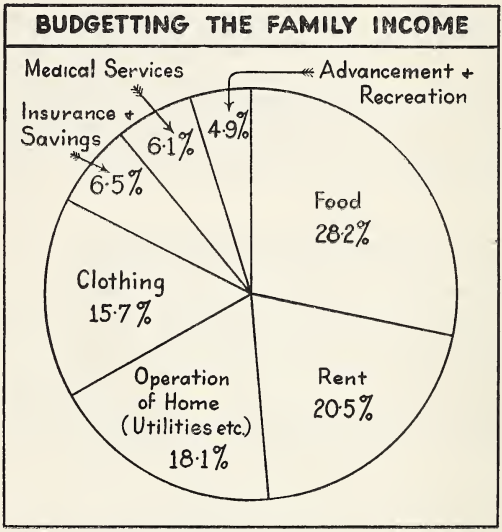
The Evolution of the Graph

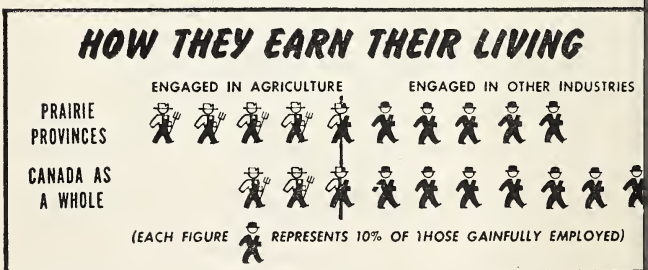
The first graphs go back to man's primitive past. Since the awn of history, he has found satisfaction and conveyed useful nformation by telling stories through the medium of pictures. he drawings in the tombs of ancient Egypt and in the caves of pain, and the pictorial writing of our North American Indians re examples. In our own day, the popularity of many illus- ated magazines underlines the appeal of pictures to both old nd young.

While graphs find their rudimentary beginnings in picture- riting, they have undergone subsequently such development nd refinement that they are now an effective instrument of athematical thinking. For example, graphs can be very useful i throwing light on mathematical laws that govern sets of data.

Most people find tables of statistical figures rather dry. The cts contained in such tables can often be made vivid and teresting by representing them on a graph. The figures on age 22 show three ways of doing this: the rectangular dis- ibution graph, the circular distribution graph or pie-graph, and ne pictograph are familiar to you. The pictograph is a link with an's early behaviour and is the forerunner of the bar graph— ne of the most common methods of depicting data. Since the






HOW THEY EARN THEIR LIVING
ENGAGED IN AGRICULTURE
ENGAGED IN OTHER INDUSTRIES
PRAIRIE PROVINCES
CANADA AS A WHOLE
(EACH FIGURE REPRESENTS 10% OF THOSE GAINFULLY EMPLOYED)

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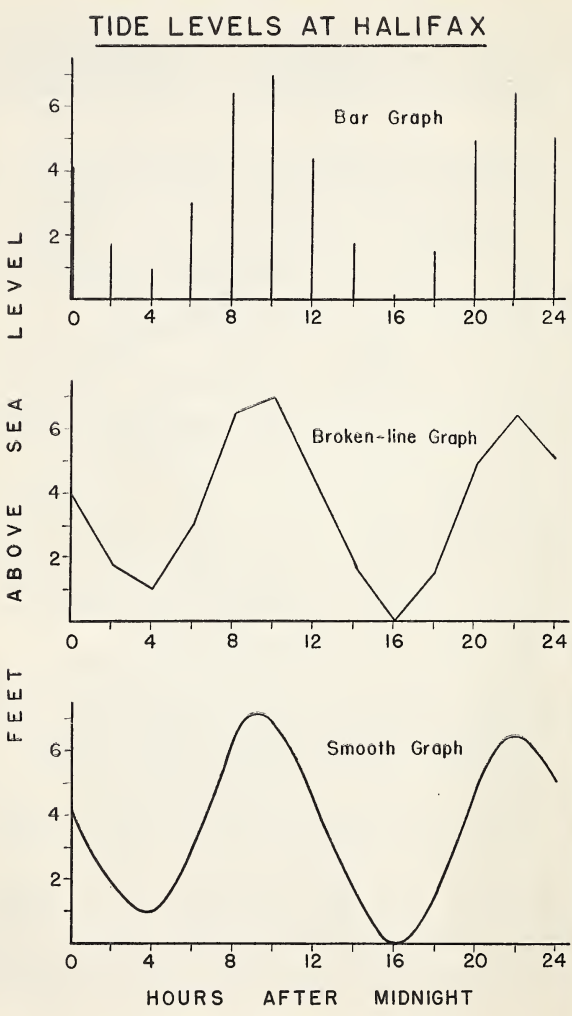
ectangle is much easier to draw than the more elaborate picture-
orms, it is natural to use rectangular bars in place of them.

The three figures on page 24 show a considerable refinement
n the making of graphs. Each shows the changes in tide levels
t Halifax over a period of time. The bar graph is easily con-
ructed and shows the relation of the tide levels to the time quite
early. However, since the important aspect of each bar is its
eight, we can get just as much information by marking the top
oints of the bars and omitting the rest. If the top points are
nen joined by a broken line, the changes stand out even more
early. This leads to the broken-line graph.

The final improvement in the graph is suggested when we
ealize that there is a definite tide level for every point of time on
ne time-axis, and not merely for those times at which obser-
ations were made. Moreover, these changes in tide levels must
ake place continuously, that is, with no sudden jumps. What
aw, if any, determines these intermediate tide levels? It is
atural and simple to assume that *a smooth curve drawn through
e points* suggests the law. Such a generalization must of course
e justified by common sense and the type of data represented.
cientists repeatedly generalize in this way, assuming that the
aw obeyed by their data is followed in other instances; they
elieve that nature does not behave erratically.

It should be noted that not all broken-line graphs lend them-
lves to this smoothing-out. If there is no meaning for values
etween those represented on the horizontal axis, then the
noothing of the graph is without purpose. For example, sup-
ose a bar graph shows the number of immigrants coming to
anada each year during the period 1930–1940. Then it would
e meaningless to smooth off the curve in order to find the
umber of immigrants entering in the year $1937\frac{3}{4}$.

When there is a meaning for values between those plotted
nd when such a value is found from the smooth graph, it is said
o be found by ***interpolation.*** The smooth graph exemplifies
ne way in which graphs may be used to find a law connecting
umerical data and thus discover new and useful information.

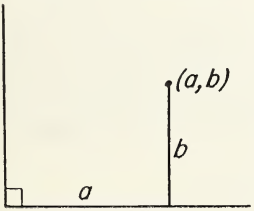

TIDE LEVELS AT HALIFAX
Bar Graph
Broken-line Graph
Smooth Graph
FEET ABOVE SEA LEVEL
2
4
6
0
4
8
12
16
20
24
HOURS AFTER MIDNIGHT

nce in this course we shall be mainly interested in smooth aphs, it will be well for us to fix attention on some of their asic characteristics.

(1) *Smooth graphs are made from two sets of related numbers.* These numbers may be obtained by measuring related quantities and tabulating the results. We shall presently see that such sets of numbers are also obtained from algebraic equations.

(2) *One number of a related pair is represented by distances measured with a suitable scale on an axis which we may take to be horizontal; the other number of the pair is represented by distances measured with a suitable scale in a direction perpendicular to this, which we may take to be vertical.*

(3) *These two distances locate a point on the graph.* The distances are called the ***coordinates*** of the point which is indicated by the notation (a, b) where a units is the horizontal distance and b units, the vertical distance. Thus each related number-pair fixes a point on the graph.



(4) *The points corresponding to the number-pairs are joined by a smooth curve. This curve illustrates the mathematical law upon which the set of data rests.*

Hence from the smooth graph of tide levels at Halifax age 24), we can read off the tide levels at 7 a.m., 1 p.m., and 30 p.m. The accuracy of these interpolations depends on the liability of the given data, the care with which the graph was ade, and the soundness of our assumption in smoothing the urve.

In smooth graphs, there is a meaning for the two numbers ssociated with each point of the curve. Graphs possessing this

property are sometimes called ***locus graphs*** (Latin: *loc* place or position; plural, *loci*.). We may think of them as t paths on which are found all points that obey certain laws; f example, the law found for a table of number-pairs. The id of locus is of great importance in mathematics and deserv special consideration.

Locus

Consider a moving point. To fix the ideas, you might imagi a small fly or a speck of chalk on a moving object. If the po moves at random, it will trace out a haphazard path with interest for us. However, should the point move under the c straint of some law, then the path traced may be very interesti indeed.

If, for example, the point is on the rim of a wheel whi rotates without rolling, a circle is traced out. When a wheel r in a straight line, the point at the centre of its hub traces a strai line parallel to the ground at a distance equal to the radius of t wheel. In each case, the path traced out is called ***the locus the point.*** Note carefully:

A path traced by a moving point is a locus.

Thus the point on the rim of the wheel mentioned above mo in accordance with the following law: The point stays at a fix distance (the radius of the wheel) from a fixed point (the cen of the wheel). What law does the point at the centre of the h obey as the wheel rolls?

The locus of a point is the path traced by the poin when it moves in accordance with a given law.

xample. A wheel rolls along a flat surface in a straight line. escribe the locus of (*a*) a point on its rim, (*b*) a point at the ntre of a spoke.

If you find it difficult to visualize these paths mentally, perform an experiment. Put chalk marks on a circular disc and roll it some distance, watching the chalk marks carefully. You can often see these loci by observing marks on the wheels of cars as they go by. In each case, the moving point partakes of two kinds of motion; rotation with the wheel and a forward motion because of the rolling. The following figures show the nature of the loci.

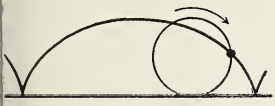


Fig. a

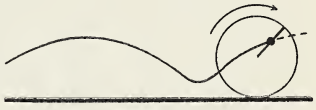


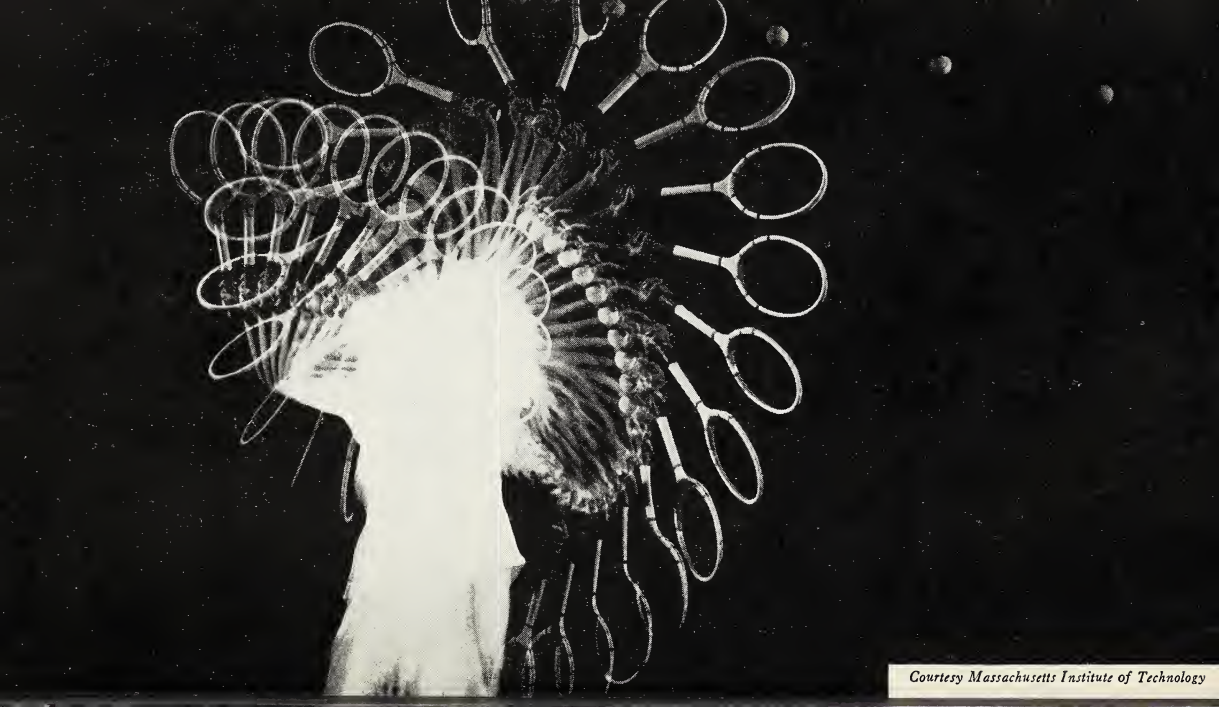
Fig. b

The curve of (*a*) is called a ***cycloid***; that of (*b*) is a sort of retched out cycloid, called a ***prolate cycloid.*** If you are terested in finding out more about these loci, read *Mathematics nd the Imagination*, by Kasner and Newman, pp. 194–200.

EXERCISES

With the aid of diagrams, describe the loci for the following ws:

1. A point moves so that it is always on the blackboard and inches from the sill.
2. A point moves on the top of (*a*) a square table, (*b*) a cir- ılar table, always keeping 2 inches from the nearest edge.
3. A point remains at the bottom of a pendulum as it swings.
4. A fly walks with uniform speed from the bottom to the)p of a pendulum as it swings.
5. A point is on a rotating propeller of an aeroplane (*a*) at st, (*b*) flying in a straight line.
6. A small object is thrown forward and upward from the)p of a precipice.



Courtesy Massachusetts Institute of Technology

7. As the turntable of a phonograph rotates uniformly, a small insect starting at the rim keeps crawling always towards the centre of the table.

8. The wheel of a freight car has a flange which projects below the surface of the track to keep the wheel from leaving the rail. What is the locus of a point on the rim of the flange as the wheel rolls on the rail?

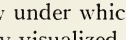
9. Horses on merry-go-rounds have an up-and-down motion as the machine rotates. Describe the motion of a point on a horse when the machine is in operation.

10. Describe the path of a baseball which is thrown into the air at an angle of about 45° with the horizontal.

11. Describe the path of a bomb dropped from an aircraft flying horizontally.

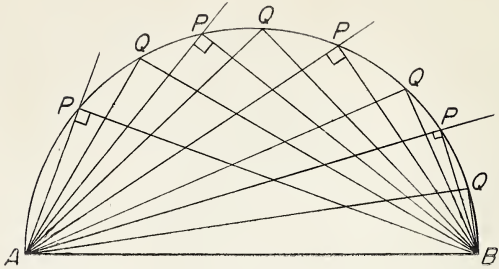
Plane Loci Found with the Aid of Instruments

Sometimes the law under which the point moves is such that the locus is not readily visualized. Many of the loci of geometry are of this nature. In such cases, it is often a good idea to adopt the method used in the locus graph: find a number of points obeying the law and assume the nature of the path. In the locus graph, the given numerical data provide us with the required points; in geometry, instruments may sometimes be used to locate a sufficient number of points to suggest the nature of the locus.



Example. Given a fixed line AB and a moving point P. If P is always at the intersection of two perpendicular lines through A and B, what is its locus?

Locate a number of positions of P as follows: Draw a number of lines through A and drop perpendiculars upon each of them from B, using a set square. The various positions of P suggest a semicircle on AB as diameter. Since the same process can be repeated below the line AB, the locus of P seems to consist of a circle on AB as diameter (page 30).



Once we have satisfied ourselves that all points P which obe the law lie on this circle, there remains an important question t be answered: *Do all points on this circle obey the law of the locus?*

All points satisfying the law lie on the locus. All points lying on the locus satisfy the law.

This latter question cannot be answered finally without callin upon a theorem of geometry, but our instruments again provid a fairly convincing answer. Draw a circle on AB as diamete take a number of points Q on the circumference, and join QA an QB for each point Q. Measure the angles AQB. What do yo observe? Does it appear that all points on the circumference ar at the intersection of two perpendicular lines from A and B? Is this true for the points A and B?

Since it now appears that

(1) all points obeying the law are on the circle having A as diameter, and

(2) all points on this circle except A and B obey the law it is a plausible conclusion that the locus of P is a circl with AB as diameter (excluding A and B).

NOTE: In order to avoid the error of rash generalization, we should b careful to take a good sampling of points P and Q, selected in a wid variety of positions.

EXERCISES

The loci in these exercises are to lie in a plane.

Find the locus of the moving point in each of the following exercises by locating a number of points with instruments. Make clear diagrams.

1. The point P moves so that it is always 1 inch from the fixed line-segment AB.

2. The point P moves so that it is always equidistant from the two fixed points A and B.

3. The point P always stays equidistant from the arms of the angle BAC.

4. A and B are two fixed points. P moves so that $PA = 2PB$.

5. AB is a fixed line-segment. P moves so that the angle $APB = 60°$.

6. AB is a fixed line and C is a fixed point one inch from AB. P moves so that it is always equidistant from the fixed line AB and the fixed point C.

7. AB is a fixed line 3 inches long. P moves so that $PA + PB = 5$ inches.

8. AB is a fixed line 3 inches long. P moves so that $PA - PB = 2$ inches.

ORAL EXERCISES

If the moving point is allowed to travel in three dimensions, try to visualize and describe the loci for the following laws:

1. P moves so as to remain 2 inches from the fixed point C.

2. P moves so that it is always 1 inch from the fixed line-segment AB.

3. P moves so that it is always 1 inch from the circumference of a fixed circle with radius 2 inches.

4. P is always equidistant from the two fixed points A and B.

Equations Regarded as Laws

We have seen that a smooth graph may be viewed in two way
(1) as the curve made up of those points, and only tho points, which satisfy a certain law; or
(2) as the path traced by a point which moves in accordan with some law.

Various points on a locus may be located by reference to the la This law may take the form of a statement in English; points ma then be located by inspection or with the aid of instrument Alternatively, the law may be implied in a table of number points are then plotted by using from the table pairs of relate numbers called the ***coordinates*** of the point. These c ordinates are used with reference to two perpendicular lin chosen as axes.

Every related number-pair gives a point on the graph or locus every point on the graph or locus gives rise to a relate number-pair.

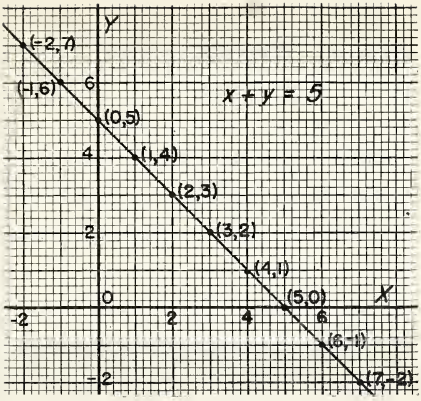
An algebraic equation like

$$x + y = 5 \quad . \quad . \quad . \quad (1)$$

may be regarded as a law satisfied by all number-pairs who sum is 5. Some pairs of numbers obeying law (1) can be arrang in a table thus:

x	-650	-50	-4	-3	$-2\frac{3}{4}$	-1	0	1	2	3.5	4	5	6	100	617.
y	655	55	9	8	$7\frac{3}{4}$	6	5	4	3	1.5	1	0	-1	-95	-612.

It is evident that if we choose a horizontal line for our x-axis a a vertical line for our y-axis and introduce suitable scales, th each pair of numbers in the table can be plotted to locate a poi We simply use the value of x for the x-coordinate and the c responding value of y for the y-coordinate. For example, t pair $x = 4$, $y = 1$ gives the point (4, 1). The completed gra is shown in the accompanying figure.



Letter-numbers like x and y of (1) which stand for any one of a large set of numbers are called ***variables.*** (See p.11.)

Note that the variables x and y of (1) are *related* since the law ys their sum is always 5; either x or y can be chosen at pleasure, ıt not both. One is said to *depend* on the other. Thus *an al-braic equation involving two variables may be regarded as a ıtement of a law giving the relation connecting the variables.*

For the purposes of algebra, the equation is the most con-ıient form for the law of the locus. From the equation, we ın readily get a table of corresponding number-pairs by taking ıe number and finding the related number by substitution. oints whose coordinates are whole numbers are easier to plot, ıt it is often necessary to use fractions in constructing a graph.

Formulas regarded as laws. If a formula is an equation two variables, we can obtain its graph as follows:

(1) *Use the formula to construct a table of values of the related numbers.*

(2) *On a piece of graph paper, draw a horizontal and a verti axis; label each axis with the variable to be measured alo it. In drawing the axes, have due regard for the values the table to be plotted. If there are no negative values, it wasteful to have large negative regions on the axes.*

(3) *Choose scales on the axes so as to make use of as much the graph paper as possible and still permit the plotting the desired values in the table.*

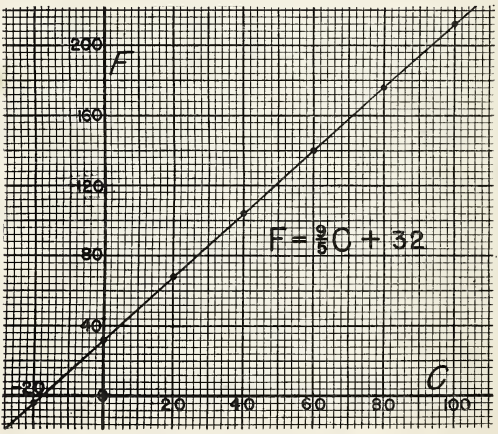
(4) *Plot the points using the corresponding number-pairs coordinates.*

(5) *Draw a smooth curve through the points.*

(6) *Label the graph with its equation.*

Example 1. A temperature of $C°$ Centigrade is the same as o of $F°$ Fahrenheit where $F = \frac{9}{5}C + 32$. Construct the graph this formula for values of C from -20 to 100.

The table and the graph are constructed as shown.



C	-20	-10	0	10	20	40	60	100
F	-4	14	32	50	68	104	140	212

Note that, in selecting scales, it is advisable to choose a aller scale for the F-axis since the values of F are larger than ose of C. The drawing of the graph is only part of the story; is also necessary to be able to read its messages. Note the lowing:

(1) Corresponding values of F and C can be read from the graph for any values within the ranges plotted. For example, if $C = 45$, $F = 113$; if $F = 100$, $C = 38$, approximately. Corresponding temperatures on the two scales are thus readily determined.

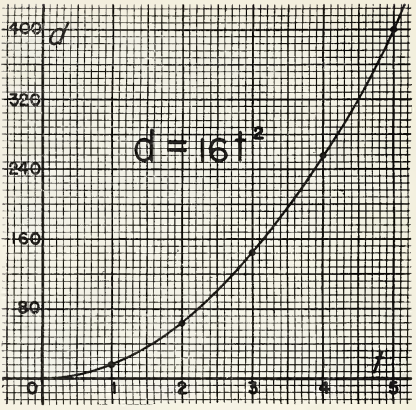
(2) As a point travels along the graph from left to right, it is moving in the direction of increasing values of C. Since at the same time the graph is rising, we see that values of F are also increasing. Thus increases in C bring about increases in F. Similarly, decreases in C are accompanied by decreases in F.

(3) In order to get more precise information about changes in the values of the two variables, note that for every increase of 20 units in C there is a corresponding increase of 36 units in F. Thus F increases 36/20 or 9/5 times as fast as C.

(4) C is negative for all values of F below 32; F is negative for all values of C below about -18.

ample 2. If a stone is dropped from the top of a cliff, the stance fallen in t seconds is d feet, where $d = 16t^2$. Draw a aph of this formula for values of t from 0 to 5.

The table of values and graph are as shown.



t	0	1	2	3	4	5
d	0	16	64	144	256	400

Once again it is best to choose a smaller scale on the d-i to allow for the greater rate of growth of d. The graph sh that

(1) as t increases, d increases slowly at first and then n and more rapidly;

(2) when the value of t is doubled, the value of d is increa 4-fold;

(3) corresponding values of t and d can be estimated for point in the ranges plotted.

EXERCISES

1. A car travelling at an average speed of 30 m.p.h. cove miles in t hours where $d = 30t$. Draw the graph of this form for values of t from 1 to 8.

2. Use the graph of Ex. 1 to answer the following questions: at distance is covered in 4.3 hours? How long does it take to el 115 miles? What change takes place in d as t increases? at change is produced in d by a change of 1 unit in t?

3. The circumference of a circle is given approximately by $6.3r$. Draw the graph of this formula for values of r from 10.

4. Use the graph of Ex. 3 to answer the following: What is circumference of a circle whose radius is 3.4 units? What is radius of a circle whose circumference is 50 units? How does ange as r decreases? What change is produced in c when r ncreased by 1 unit? Assuming that the earth is a perfect ere, how much longer than the equator is a circle surrounding earth one foot above the equator?

5. The area of a spherical ball is given approximately by $= 88r^2/7$. Construct the graph of this formula for values of om 0 to 5.

6. Use the graph of Ex. 5 to describe the behaviour of A as r eases. What is the radius of a ball whose area is 200 units?

7. A father was 24 when his son was born. Denoting the 's age by x years and the father's age by y years, write the nula which gives the relation between their ages. Construct graph of this formula for values of x from 0 to 10.

8. In making a journey of 360 miles a man has a choice of ious means of transport, with speeds ranging from 10 m.p.h. 50 m.p.h. Draw a graph to show the relation between the ed and the time taken on the journey. Write the formula ch gives this relation.

9. One inch equals about 2.54 centimetres. Construct a ph from which one could read off the number of centimetres ny length up to 10 inches. What formula gives the relation ween inches and centimetres? Could this graph be used to l the number of inches in 10 centimetres?

10. If the water in a tank is h feet deep, then when a tap at bottom of the tank is opened the water flows out with a speed

of v ft. per sec. where $v = 8\sqrt{h}$, approximately. Construc graph to show the relation between v and h.

11. Use the graph of Example 1 on page 34 to find the tem ature whose measure is three times as much on the Fahren scale as on the Centigrade scale. Compare the measure of temperature on the Centigrade scale with the value of x obtai by solving

$$3x = \tfrac{9}{5}x + 32.$$

Graphs of First-Degree Equations

In a previous course, you plotted a number of equation the first degree in x and y. (Although the letters x and y are u here and later for definiteness, they could be replaced by o letters.) The results led to the following tentative conclusi

> ***The graph of an equation of the first degree in x and y is always a straight line.***

This fact was not actually proved but rather made acceptable the graphs drawn for many special cases. You are now aske accept the fact that this theorem can be completely demonstra by methods which we shall not go into at this time. Thus i quite natural that first-degree equations in x and y are of called ***linear equations***. In an equation like

$$3x - 2y = 6,$$

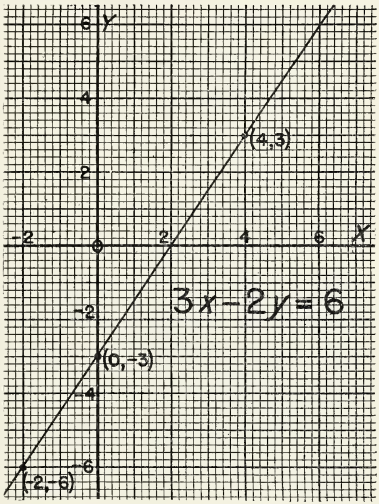
recall that (1) every pair of values of x and y that satisfy equation give a point (x, y) on a straight line, and (2) ev point on this straight line has coordinates that satisfy the eq tion.

Example. Draw the graph of $3x - 2y = 6$.

Since we know the graph is a straight line, it is sufficient locate two points. It is advisable, however, to locate a th point as a check. We get the following table:

$$3x - 2y = 6$$

x	-2	0	4
y	-6	-3	3



graph is shown in the figure.

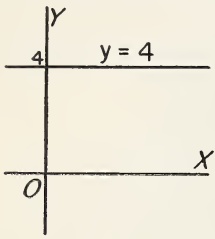
f possible, it is best to use whole numbers for the plotting. , it is a good idea to get points not too close together. Why?

E: Just as in formula graphs, the locus should be regarded omething more than plotted points. It is instructive always call that the graph may be regarded as the path traced by moving point (x, y) as this point moves in obedience to the $3x - 2y = 6$. As we imagine (x, y) to travel along the line left to right, we see that as x increases, y increases. More- , when x increases by 1 unit, y increases by 1.5 units. It is

also evident from the graph that y is positive, zero, or nega
according as x is greater than 2, equal to 2, or less than 2.

Example. What is the graph of $y = 4$?

Such graphs can be seen by inspection, if we regard the



the paths of points moving u
law. In this case, since (x, y)
move so as always to keep its y-
dinate equal to 4, the locus
straight line parallel to the x
and 4 units above it.

EXERCISES

What path is traced by the moving point (x, y) as it mov
accordance with each of the following conditions?

1. $x = 2$.	**2.** $x + 4 = 0$.	**3.** $y = -5$.
4. $y = 0$.	**5.** $x = 0$.	**6.** $2y = 7$.
7. $3x = -4$.	**8.** $x = y$.	**9.** $x = -y$.

Draw the graph of each of the following linear equations:

10. $x + y = 6$.	**11.** $y - x = 5$.
12. $x - 2y = 0$.	**13.** $2x + y = 2$.
14. $x - 3y = -3$.	**15.** $3x - y = -6$.
16. $3x - 4y = 7$.	**17.** $5x + 2y = -9$.

18. Use the graph of Ex. 11 to find (*a*) the value of y v
$x = 3$, (*b*) the value of x when $y = -1$.

19. Use the graph of Ex. 12 to answer the following: (*a*)
does y change as x increases? (*b*) What change is produced
by an increase of 1 unit in y?

20. Use the graph of Ex. 13 to answer the following: (*a*)
does x change as y increases? (*b*) What change is produced
by an increase of 2 in x?

21. State in words the condition which the moving point
satisfy as it traces the graph of (*a*) Ex. 16, (*b*) Ex. 17.

Graphs of Linear Systems

Since each equation of the first degree in x and y gives rise to raight-line graph, it follows that the graph of a linear system wo equations consists of two straight lines. Two lines can be ted to each other in three different ways: they may meet in point, they may meet in no point, or they may coincide. figure illustrates the three possible graphs.

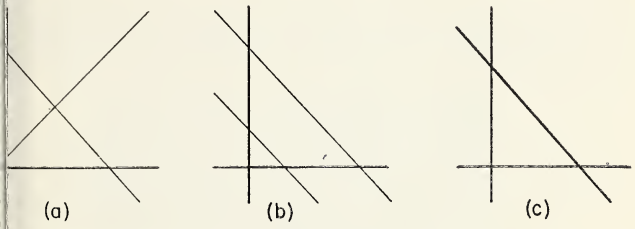


Fig. (a) shows two lines that have exactly one point in mon. Therefore, there is only one number-pair that satisfies h equations of the corresponding linear system. Such a system id to have a *unique solution*. Since the graphs in (b) have no t in common, there is no number-pair that satisfies both ations of the system. The two equations corresponding to are said to be *inconsistent*. Finally, in (c) the graphs coincide; any point on the graph of the first equation is also on the ph of the second, and its coordinates satisfy both equations. system has therefore an unlimited number of solutions; its equations are equivalent.

system of two linear equations in two unknowns may have solution, no solution, or an unlimited number of solutions.

When the solution of a linear system is found by plotting phs of the two equations, your answer may be only approxiely correct. If the correct solution involves fractional values and y, your graph paper often enables you to make a good

estimate and no more. When the graphs intersect on the co of a square of the graph paper, we can find the solution curately.

While the graph of a linear system throws light on its nat it is usually not the best way to find the solution of the syst if any. Algebraic methods are in most cases preferable.

EXERCISES

Draw graphs of the following systems of linear equati For each system, tell whether the equations have one solut no solution, or an unlimited number of solutions. Give uni solutions where they exist.

1. $x + y = 2$, $2x + y = 3$.

2. $x - y = 1$, $x + y = 3$.

3. $x - y = 0$, $x - 2y = 0$.

4. $x + y = 1$, $y = -2$.

5. $x + 2y = 2$, $x + 2y = 4$.

6. $2x - y = 4$, $x = 3$.

7. $x + 3y = 3$, $3x - y = 9$.

8. $2x + y = 4$, $4x + 2y = 8$.

9. $4x - y =$, $6x + 3y =$

10–18. Verify the solutions found in Exs. 1–9 by solving systems algebraically.

19. Show by a graphical method that there are no value x and y that will satisfy the three following equations:

$3x - 5y = -4$, $x + 2y = -5$, and $x - y = 1$

Some Special Types of Graph

Most of the equations considered so far in this chapter h involved x and y only to the first power. When second or hig powers appear, the graphs will in general become more comp Certain kinds of equations containing terms with second po of x and y are easily plotted.

Example. Plot the equation $x^2 + y^2 = 25$.

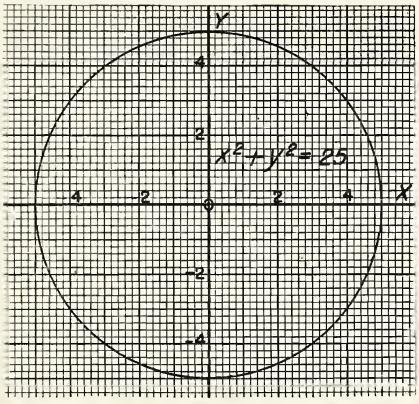
Construct a table in the usual way. For example, if x then the equation gives $y^2 = 25$ whence $y = 5$ or -5. T

we have two points, (0, 5) and (0, −5), on the graph. Similarly, if $x = 1$, $y^2 = 24$, whence $y = \pm\sqrt{24}$ or ± 4.9. (See table of square roots on page 325.)

Continuing in this way, we get the following table and graph.

$$x^2 + y^2 = 25$$

x	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
y	—	0	±3	±4	±4.6	±4.9	±5	±4.9	±4.6	±4	±3	0	—



Note that for values of x numerically greater than 5, y^2 becomes negative so that there is no value for y. This indicates that there are no points on the graph with x-coordinates greater than 5 or less than −5.

EXERCISES

ing the method of the last example, draw the graph for each the following equations:

1. $x^2 + y^2 = 9$. 2. $x^2 + y^2 = 36$. 3. $x^2 + y^2 = 16$

The drawing of the foregoing three graphs is sufficient to attention to the fact that they have something in common. N

(1) Each graph is a circle with its centre at the origin.
(2) The radius of each circle is the square root of the nu on the right side of the equation.

It should therefore occasion no great surprise to know tha following theorem can be proved:

> *The graph of the equation* $x^2 + y^2 = r^2$ *is a circle with centre at the origin and radius* r.

This fact enables us to draw the graph of an equation $x^2 + y^2 = 49$ without plotting any points. Simply draw a c with centre at (0, 0) and radius 7. For equations $x^2 + y^2 = 13$, in which the right side is not a perfect sq you can either get the radius from a table of square roo else try to guess one point on the circle. For example, (2, 3) satisfies $x^2 + y^2 = 13$, the radius is the distance (0, 0) to (2, 3). When the radius can be found in this way avoid the plotting of fractions.

Example. Draw the graph of $y = x^2 - 2x$ for values of x -4 to 6.

Substitution in the usual manner gives the following t

$$y = x^2 - 2x$$

x	-4	-3	-2	-1	0	1	2	3	4	5	
y	24	15	8	3	0	-1	0	3	8	15	2

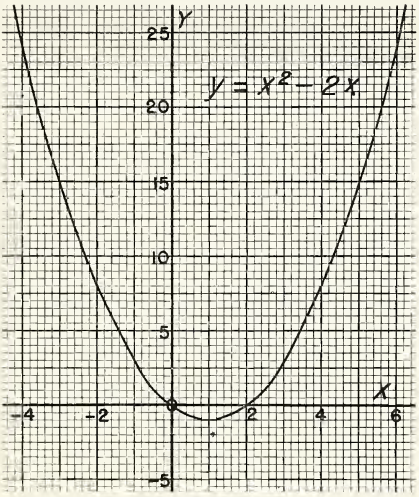
In setting up the graph, it is well to note the following:

(1) Most of the values of y are larger than the correspon values of x. Hence, let 1 inch on the x-axis repre 2 units, and 1 inch on the y-axis represent 5 units.

2) The values of y in the table "balance" about -1 which occurs when $x = 1$. Therefore in order to have the graph appear centred on the page, see that the centre vertical line of the graph paper is at $x = 1$.

3) Since values of y do not become smaller than -1, the x-axis is taken low on the page. We do not need a large negative range on the y-axis.

The completed graph is shown in the figure.



EXERCISES

1. Use the graph of the last example to answer the following stions: What values of x make $y = 0$? What values of x e $y = 8$? What is the smallest value y can have? What es of x make y (*a*) negative? (*b*) positive? For what values does y increase as x increases? Estimate the values of x make $y = 10$.

Draw the graph of each of the following equations:

2. $y = x^2 - 4x$, for values of x from -4 to 8.
3. $y = x^2 + 2x$, for values of x from -7 to 5.
4. $y = -x^2 + 2x$, for values of x from -4 to 6.

Summary

The ideas of the chapter, though not all new, are of siderable importance. The salient facts are as follows:

(1) *An algebraic equation in x and y may be regarde* ***a law which determines the positions of variable point (x, y).***

(2) *The graph of an equation in x and y is the path trac* (x, y) *as it moves under the law. This graph is also c* ***the locus of the equation.***

(3) *The graph can be plotted by using the equation to cons a table of corresponding number-pairs which satisf equation.*

(4) *Each number-pair that satisfies the equation locates a* (x, y) *on the graph.*

(5) *Each point on the graph has coordinates that satisf equation.*

(6) *The graph of the equation provides a vivid picture manner in which changes in one variable bring changes in the other.*

EXERCISES ON CHAPTER II

A

1. Name six kinds of graphs.
2. What is meant by the ***coordinates*** of a point?
3. What is ***the locus of a point***?
4. What is the locus of a point on a doorknob as the swings open?
5. What is the locus of a point which moves in three di sions so that it always remains equidistant from every p on the circumference of a fixed circle?

5. What is a ***variable***?

7. What is a common method of expressing a relation between variables?

8. What is a common method of writing a law governing motion of a moving point (x, y)?

9. Why is an equation of the first degree in x and y sometimes ed a ***linear equation***?

0. How many points should be plotted in order to draw the ph of a linear equation? Draw the graph of $x - 2y = -2$.

1. What are the possibilities for solutions of a system of two ar equations? Write two linear equations that are in- istent.

B

w graphs of the following equations:

2. $x + 5 = 0$.

3. $3x - y = 3$.

4. $x^2 + y^2 = 1$.

5. $x^2 + y^2 = 5$.

6. $y = x^2 - x$.

13. $x + 2y = 0$.

15. $5x - 3y = 15$.

17. $x^2 + y^2 = 64$.

19. $y = x^2 - 4$.

21. $y = -x^2 - 2x$.

22. The area of a circle is given approximately by the formula $A = 3.14r^2$. Construct a graph of this formula for values of r from 0 to 6.

Draw graphs of the following systems of linear equations. e unique solutions where they exist.

23. $x - y = -1$
$x + 2y = 8$.

24. $3x - y = 3$
$3x - y = 0$.

25. $x - 2y = 6$
$2x - 3y = 11$.

26. $x - 3y = 3$
$2x - 6y = 6$.

27. Use the graph of Ex. 22 to answer the following: What nge is produced in A as r increases? What is the radius of a e whose area is 50 units?

28. Use the graph of Ex. 14 to answer the following: What nge occurs in y as x increases? What change is produced in y n increase of 1 unit in x? For what values of x is y positive?

29. Use the graph of Ex. 20 to answer the following: (*a*) W values of x make $y = 0$? (*b*) What values of x make y (*c*) What is the smallest value y can have? (*d*) For what v of x is y negative?

30. Use the graph of Ex. 21 to answer the following: (*a*) W values of x make $y = 0$? (*b*) What is the greatest value y have? (*c*) What value of x makes y take its greatest v (*d*) For what values of x is y positive?

31. AB is a fixed line 4 inches long. P moves so that P $PB = 6$ inches. Find the locus of P by locating a numb points with instruments.

32. AB is a fixed line 3 inches long, and P moves so $PA = 3PB$. Find the locus of P by locating a number of p with instruments.

C

33. Find the coordinates of the point where the grap $4x - 3y = 12$ cuts (*a*) the x-axis, (*b*) the y-axis, (*c*) the g of $x + 2y = 14$.

34. Solve graphically:
$$2x - 3y + 5 = 0$$
$$4x - y - 5 = 0.$$

35. Solve graphically:
$$x^2 + y^2 = 25$$
$$x - y = 1.$$

36. Solve graphically:
$$y = x^2$$
$$x - y + 2 = 0.$$

37. A degree of latitude on the earth is approximately 69 m Construct a graph from which one could read off the distan miles of a place from the equator when the latitude of the is known. Could this graph be used to find the latitude place whose distance from the equator is known?

38. (*a*) Represent graphically the equations $x - 3y + 5$ $2x + y + 3 = 0$. (*b*) Find the coordinates of the point of i section of the graphs of (*a*) and show that this point is o graph of $3x - 2y + 8 = 0$.

9. Draw the graph of the equation $y = x^2 - 5x + 4$.

0. Draw the graph of the equation $y = -x^2 + x + 6$.

1. Use the graph of Ex. 39 to answer the following: What ues of x make $y = 0$? What values of x make y negative? at is the smallest value y can have? For what values of x does crease as x increases?

2. Show graphically that there is one number-pair that sfies all three of the equations $2x + 3y = 12$, $3x + 2y = 13$, $x + 4y - 11 = 0$.

3. The volume of a spherical ball is given approximately by $= 4.19r^3$. Construct a graph of the formula for values of r n 0 to 5. Describe the behaviour of V as r increases. What ie radius of a ball whose volume is 200 units?

4. A boy swims across a stream, always heading for the rest point on the opposite shore. Describe his path if (*a*) the am has a uniform current, (*b*) the stream runs faster in the dle than along the edges.

5. A duck swims across a stream, always heading for the t on the opposite bank directly opposite her starting point. cribe the duck's path if the stream has a constant current ch is (*a*) less than, (*b*) equal to, (*c*) greater than the duck's d in still water.

CHAPTER III

RATIO, PROPORTION, AND VARIATION

The Ratio of Two Numbers

Suppose that you wish to compare two numbers such 6 and 2. Two simple methods are as follows:

(1) You may say that 6 is 4 ***more*** than 2. This is c parison by ***subtraction.***

(2) You may say that 6 is 3 times 2, or that 2 is 2/6 or of 6. This is comparison by ***division.***

The second method is the one of greater interest to us at pre When we compare two numbers by dividing one by the o we say that we have found the ***ratio*** of the two numbers.

Examples. The ratio of the two numbers 8 and 12 is 8/12 or and the ratio of 25 and 20 is 25/20 or 5/4.

Thus, ***the ratio of two numbers is the fraction obta when the first number is divided by the second.***

> *The ratio of* a *to* b *is* $a \div b$, *or* a/b. a/b *is also written* $a{:}b$, *where* a *and* b *are called the* ***terms of the ratio.***

Since ratios are fractions, it follows that they are subje the same principles that apply to fractions. Hence,

$$\frac{a}{b} = \frac{ma}{mb}.$$

Therefore, *if the terms of a ratio are multiplied or divided b same number,* ($\neq 0$), *the value of the ratio is unchanged.* example,

$$2/3 = 4/6 = 6/9 = 8/12 = 10/15 = \quad . \quad . \quad .$$

Note that if we are given two numbers in the ratio 15:19, we ***not*** justified in concluding that the numbers are 15 and 19. ere is no end to the number-pairs that have the ratio 15:19; example, 30/38, 150/190, 45/57, 15000/19000, and so on. ncerning two numbers in the ratio 15:19, we have the fol- ing facts:

(1) The first number divided by the second equals 15/19.
(2) The smaller number is 15/19 of the larger, and the larger is 19/15 of the smaller.
(3) The numbers can be represented by $15x$ and $19x$, where x is any number other than 0.

mple. Divide 165 into two parts in the ratio of 4:7.
This problem can be solved in a number of ways. The following three methods are worth noting.

Method 1. Let the parts be x and $\frac{4}{7}x$.
Then $x + \frac{4}{7}x = 165$.
Hence $7x + 4x = 1155$,
and $11x = 1155$, whence $x = 105$.
Therefore the parts are 105 and 60.
Verification: $60 + 105 = 165$, and $60/105 = 4/7$.

Method 2. Let the parts be $4x$ and $7x$.
Then $11x = 165$, whence $x = 15$.
Therefore the parts are 60 and 105, as before.

Method 3. Since 165 is to be divided in the ratio of 4:7, we may think of the whole as consisting of $4 + 7$ or 11 equal parts. Then
the smaller part is 4/11 of 165, or 60, and
the larger part is 7/11 of 165, or 105.

EXERCISES

For oral answers

te, in the simplest form, the ratio of the first number to the ond number for each of the following pairs:

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1. 12, 24.	**2.** 15, 45.	**3.** 21, 35.
4. 55, 33.	**5.** $5x$, $7x$.	**6.** 1/2, 1/3.
7. 0.3, 0.05.	**8.** $33\frac{1}{3}$, 100.	**9.** 12.5, 100.
10. $24a$, $32a$.	**11.** 0.25, 3.5.	**12.** 3/8, 5/4.

Write the following statements in your workbook and fill the blanks:

13. If two numbers are in the ratio 4:5, the smaller numbe ..... of the larger, the larger number is ..... of smaller, the smaller is ..... of the sum of the two nu bers, the larger is ..... of the sum of the two numbers

14. If two numbers are in the ratio 7:2, the smaller numbe ..... of the larger, the larger is ..... of the sum the two numbers.

15. If x is two-thirds of y, then the ratio x/y is .....

16. If a is five-fourths of b, then the ratio a/b is ..... b is ..... of a, and a is ..... of the sum $a + b$.

17. If 210 is divided in the ratio 5:6, then

(*a*) the smaller part is ..... of the larger part;

(*b*) the smaller part is ..... of 210;

(*c*) the larger part is ..... of 210.

18. If $a = 35$ and $b = 42$, then (*a*) $a:b =$ .....; a eq ..... of b; b equals ..... of a.

For written answers

Reduce the following ratios to their simplest forms:

19. $2\frac{1}{2} : 3\frac{1}{3}$.

20. $14ab : 21a^2b$.

21. $x^2 - y^2 : x$

22. $5.25 : 6\frac{3}{4}$.

23. $34m^2n^3 : 51m^3n^2$.

24. $1 - \frac{1}{a} : 1 +$

25. Express each of the following per cents as a ratio: (*a*) 2 (*b*) 25%, (*c*) 40%, (*d*) 75%, (*e*) $66\frac{2}{3}$%.

26. Which ratio in each of the following pairs is the gre (*a*) 4:5 or 5:6? (*b*) 22:7 or 314:100? (*c*) 17:18 or 19:20?

27. Form new ratios by adding to the terms of 2:3, in cession, the numbers 1, 2, 3, 4, 5. What appears to be the e of continuing such additions?

28. Form new ratios by adding to the terms of 3:2, in succession, the numbers 1, 2, 3, 4, 5. What appears to be the effect continuing such additions?

29. What number must be added to each term of the ratio 11 to produce a ratio equal to 2:3?

30. What number must be subtracted from each term of the tio 6:13 to produce a ratio equal to 2:3?

31. Divide 360 into two parts in the ratio 5:13. (Give three utions.)

32. Divide 540 into two parts in the ratio (*a*) 5:4, (*b*) 9:11, 8:19. (Use a different method of solution for each part of this oblem.)

Ratios of Quantities

Recall that ***a quantity is a number of units of something***; example, \$50, 8.5 yards, 60 miles per hour, 125 tons are antities. What is meant by the ratio of \$50 to \$100? We nnot divide \$50 by \$100; *only numbers can be divided.* The swer is not difficult. Obviously, \$50 is one-half as much money \$100, and this ratio is obtained by dividing 50 by 100. Hence, get the ratio we divide the ***measures*** of the quantities.

In the same way, the ratio of 8 yards to 17 yards is 8/17, the io of 60 m.p.h. to 45 m.p.h. is 4:3, and the ratio of 96 lb. to lb. is 3/2. However, the ratio of 2 feet to 9 inches is ***not*** 2/9; fair comparison like quantities must be expressed in the same its. Since 2 feet equals 24 inches, the ratio of 2 feet to 9 inches 24/9 or 8/3.

> ***The ratio of two quantities of the same kind is the ratio of their measures when the quantities are expressed in the same units.***

ample 1. If 12 inches equal 30.48 centimetres, find the ratio one metre to one yard.

We express both the metre and the yard in the same uni preferably centimetres in this case. We have

1 metre = 100 cm., and 1 yard = 36 inches = 91.44 cm

Therefore 1 metre:1 yard = 100 : 91.44 = 1.09 : 1.

Example 2. Smith and Jones invest \$1500 and \$1200, respecti ly, in a business. They agree to share profits in the ratio of th investments. How should a profit of \$108 be divided?

The ratio of the investments is 1500:1200 or 5:4. Therefo let Smith receive \$5x and Jones, \$4x. We have

$$9x = 108, \text{ whence } x = 12.$$

Therefore, Smith gets \$60 and Jones, \$48.

The meaning of the ratios *a:b:c*. Of the three numbers 1 150, 200, the ratio of the first to the second is 100:150 or 2:3, a the ratio of the second to the third is 150:200 or 3:4. Both sta ments can be combined by saying that the three numbers are the ratios 2:3:4.

Similarly, if three boys have weights of 120 lb., 140 lb., a 170 lb., respectively, we can give the ratios of their weig as 12:14:17.

If three numbers a, b, c *are in the ratios* 3:4:5, *then*

$$\frac{a}{b} = \frac{3}{4} \text{ and } \frac{b}{c} = \frac{4}{5},$$

$$\text{or } \frac{a}{3} = \frac{b}{4} = \frac{c}{5}.$$

Example 3. Three countries A, B, and C agree to keep their na tonnage in the ratios 3:5:7, respectively. How should th combined total naval strength of 18,000,000 tons be apportion

Let A, B, and C have respectively 3x tons, 5x tons, a 7x tons.

Then, we have $15x = 18{,}000{,}000,$
whence $x = 1{,}200{,}000.$
Therefore A should have 3,600,000 tons, B 6,000,000 tons, and C 8,400,000 tons.

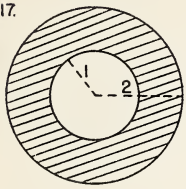
EXERCISES

In each of the following pairs of quantities, find the ratio of e first to the second. Express each ratio as a fraction in its vest terms.

1. \$75, \$105. 2. \$2, 80 cents. 3. 18 hours, 3 days.
4. 8 yards, 36 feet. 5. 3 tons, 15 cwt. 6. 220 yd., 1 mile.
7. 3 hours, 240 minutes.
8. 1 m.p.h., 1 ft. per sec.
9. 12 oz., 2.5 lb.
10. Divide \$150.30 into three parts in the ratios 4:5:6.
11. Two men enter a partnership agreeing to share the profits the ratio of their investments. If the first partner invests 500 and the second, \$3500, how should a profit of \$435.48 divided?
12. A farmer wishes to enclose a rectangular field with 224 ls of fencing. If the ratio of the length and width is 4:3, find e dimensions of the field.
13. If 12 inches equal 30.48 centimetres, find the ratio of) 8 feet to 2 metres, (*b*) 1 mile to 1 kilometre.
14. If 1 oz. equals $28\frac{1}{3}$ gm., find the ratio of (*a*) 1 lb. to kgm., (*b*) 3 lb. to 1000 gm.
15. A 300-foot chain is divided into three lengths in the tios 1:3:5. Find the length of the longest piece of chain.
16. The radius of an orange is R units before the skin is moved, and r units after. If $R:r$ equals 5:4, find the ratio of the lume of the unskinned orange to the volume of the skinned ange. What fraction of the orange is edible? What is the ratio the non-edible part to the edible part? (Volume of a sphere $4\pi r^3/3$.)

For each of the following figures, find the ratio of the shad portion to the unshaded.

17.



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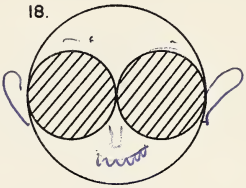

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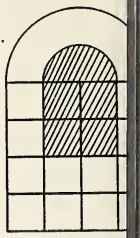


Proportion

Many simple problems give rise to sets of equal ratios. I example, if coal is $16 per ton, we have the following table.

Number of tons, n	1	2	3	4	5	6	7
Total cost in dollars, c	16	32	48	64	80	96	112

It is seen that ratios of corresponding number-pairs in the ta are equal:

$$1/16 = 2/32 = 3/48 = 4/64 = \quad . \quad . \quad .$$

The equation which states that two ratios are equal is called a ***proportion.***

Thus $1/16 = 2/32$ is a proportion. We say that the fo numbers 1, 16, 2, 32 are ***in proportion,*** or they are ***propo tionals.*** Similarly, since $2/4 = 4/8$, the four numbers 2, 4, 4 are in proportion. The number 4 has a special role here becau it occurs as both middle numbers. We call 4 the ***me proportional*** between 2 and 8.

If $a/b = b/c$, then b is a ***mean proportional*** *between a and c.*

A proportion may be written $a : b = c : d$. The four basic nciples for dealing with equations (page 8) enable us to re-ange proportions in various ways.

ample 1. If $a : b = c : d$, prove that $b : d = a : c$.

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. (Multiply by bd)

Therefore $\frac{a}{c} = \frac{b}{d}$. (Divide by dc)

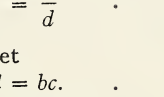
Note on cross-multiplication. When both sides of the pro-tion

$$\frac{a}{b} = \frac{c}{d} \quad . \quad . \quad . \quad (1)$$

multiplied by bd, we get

$$ad = bc. \quad . \quad . \quad . \quad (2)$$

uation (2) can be obtained from (1) by the following scheme:



rm products as suggested by the arrows, and equate the ults.

The foregoing scheme is often referred to as ***cross-ultiplication.*** It is not a new principle, but rather a device gaining speed in re-arranging a proportion such as (1) into an ıation such as (2). The effect is the same as that produced en both sides of (1) are multiplied by bd.

ample 2. If $8 : 3 = 12 : x$, find x.

If we multiply both sides of the equation by $3x$ (or cross-multiply), we have

$8x = 36$, whence $x = 36/8 = 9/2 = 4.5$.

Example 3. Find the mean proportional between 3 and 27.

Let x be a mean proportional. Then

$$3/x = x/27.$$

Hence $x^2 = 81$, and $x = 9$ or -9.

The mean proportional is therefore 9 or -9.

Example 4. What number must be added to each of the numb 2, 7, 8, 23 so that the sums will be in proportion?

Let x be added to each of the given numbers. If the sums in proportion, then

$$\frac{2+x}{7+x} = \frac{8+x}{23+x}.$$

Multiply each side of this equation by $(7+x)(23+x)$, cross-multiply, and get

$$(2+x)(23+x) = (8+x)(7+x).$$

Therefore $46 + 25x + x^2 = 56 + 15x + x^2$,

and $10x = 10$, whence $x = 1$.

The required number is 1. The answer is verified by not that $3/8 = 9/24$.

EXERCISES

In each of the following proportions, find the value of unknown number:

1. $\frac{4}{5} = \frac{a}{3}$. **2.** $\frac{-3}{11} = \frac{12}{x}$. **3.** $\frac{m}{14} = \frac{3}{-2}$.

4. $x : 4 = 3 : 2$. **5.** $8 : n = n : 2$. **6.** $24 : x = -3$.

7. $(x - 1)/3 = 7/2$.

8. $2.4/r = 3.2/2.8$.

9. $8/5 = 4/(x + 1)$.

10. Solve $m : n = r : s$ for (*a*) m, (*b*) s.

11. Solve $x : y = c : d$ for y. Find y when $x = 10$, $c = 12$, $d =$

12. Find the ratio $x:y$ given (*a*) $3x = 4y$, (*b*) $5x - 3y =$ (*c*) $7x + 2y = 0$.

13. If $ak = mn$, show that (*a*) $a : m = n : k$, (*b*) $m : k = a$

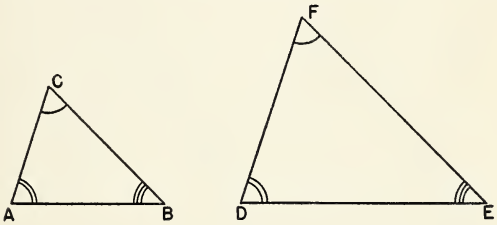
4. What number must be added to each of the numbers , 21, 37 so that the sums will be in proportion?

5. What number must be subtracted from each of the nbers 10, 14, 28, 44 so that the differences will be in pro- tion?

6. Find a mean proportional between (*a*) 4 and 16, (*b*) 0.5 l 4.5, (*c*) a^2b^3 and a^4b, (*d*) $(x + y)^2$ and $(x - y)^2$.

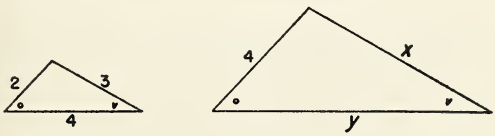
Proportions in Geometry

An interesting source of equal ratios is provided by a theorem geometry. *Two triangles are similar if the angles of one are re- ctively equal to the angles of the other.* Sides opposite such rs of equal angles are called ***corresponding.*** It is seen that



similar triangles have the same shape but not necessarily same size. The following theorem is proved in geometry:

> ***If two triangles are similar, the ratios of corresponding sides are equal.***



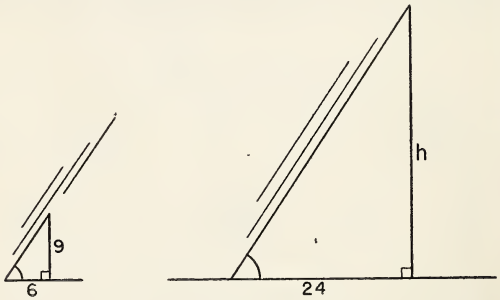
Thus, for the triangles shown, we may write

$$\frac{x}{3} = \frac{4}{2}, \quad \text{whence } 2x = 12 \text{ and } x = 6.$$

Also $y/4 = 4/2$, whence $2y = 16$ and $y = 8$.

The use of equal ratios as found in similar triangles known in ancient times. The Greeks applied their knowledg similar triangles to solve problems involving heights and tances.

Example. In order to find the height of a flagpole, an obse holds a 9-foot rod perpendicular to the ground, noting that rod casts a 6-foot shadow. If the flagpole casts a 24-foot sha at the same time, what is its height?



Since the sun's rays are parallel, the two triangles show the figure are similar. Let the flagpole be h feet tall. T from similar triangles, we get the proportion

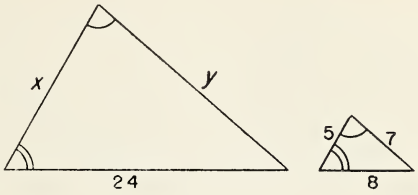
$$\frac{h}{9} = \frac{24}{6}, \text{ which gives } 6h = 216 \text{ and } h = 36.$$

Therefore, the flagpole is about 36 feet tall. The degr accuracy of the answer depends on that of the given meas ments.

EXERCISES

nd the values of x and y for each of the following pairs of similar angles:

1.



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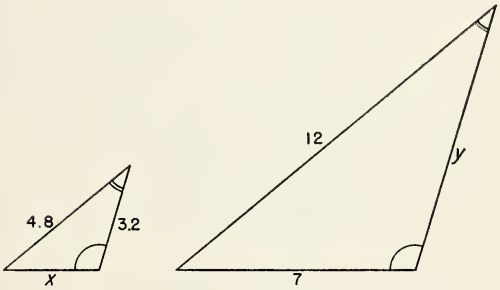


3. Mr. Jones' shadow was found to measure 88 inches at the me time as his son's shadow measured 3′ 8″. If Mr. Jones is 2″ tall, find his son's height.

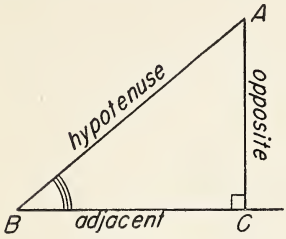
4. A tree casts a shadow 31′ 8″ long at the same time that a -foot pole, held perpendicular to the ground, casts a shadow 2″ long. How high is the tree?

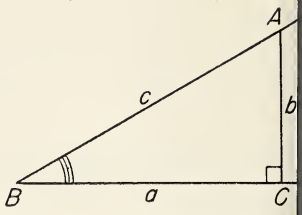
5. Describe a method by which ancient calculators might ve found the height of the Great Pyramid.

Ratios Used in Trigonometry

The subject of trigonometry deals with properties of triangles. soon as we begin to investigate triangles, we are likely to find

certain ratios coming into prominence. To study these ratio let us consider first the angle B.





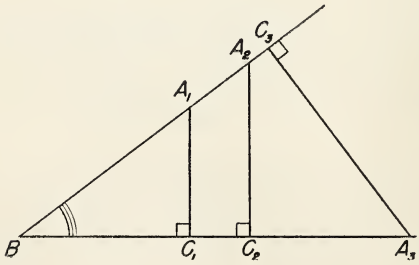
If from a point A on one of the arms of angle B a perpendicul is dropped to the other arm, we get a right-angled triangle AB The sides of this triangle are given the following names:

AC (of measure b) is the side ***opposite*** angle B.
BC (of measure a) is the side ***adjacent*** to angle B.
AB (of measure c) is the ***hypotenuse.***

Now we can get the measures of these sides and form the follo ing ratios:

$$b/c, \quad a/c, \quad b/a.$$

If we repeat the process for a different point A, we get different triangle. However, the three ratios will be found have the same values as before.



hus,

$$\frac{A_1C_1}{B\,A_1}=\frac{A_2C_2}{B\,A_2}=\frac{A_3C_3}{B\,A_3}.$$

his result can be seen without measuring the lines; we need only pply the theorem about similar triangles on page 59.

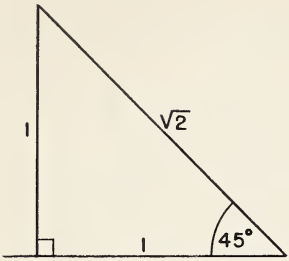
Hence the ratio of the opposite side to the hypotenuse depends ly on the angle B. This ratio is called ***the sine of angle B,*** d is written ***sin B.*** Similarly, the ratio of the adjacent side to e hypotenuse remains constant if B is unchanged. This ratio is lled ***the cosine of angle B*** and is written ***cos B.*** Finally, e ratio of the opposite side to the adjacent side is called ***e tangent of angle B*** and is written ***tan B.***

$$\textit{sin } B=\frac{\textit{opposite side}}{\textit{hypotenuse}};\quad \textit{cos } B=\frac{\textit{adjacent side}}{\textit{hypotenuse}};$$

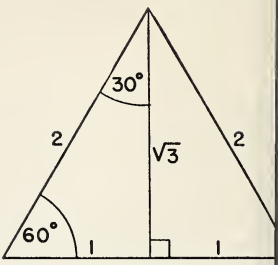
$$\textit{tan } B=\frac{\textit{opposite side}}{\textit{adjacent side}}.$$

These three ratios of an acute angle may be found approximately by construction and measurement. Such a process laborious; in order to make the ratios readily available, they ve been computed and tabulated for our use. A table giving e ratios of acute angles at intervals of 1 degree is to be found the end of the book.

Trigonometric ratios of some special angles. The trigonometric ratios of certain angles can be found exactly by simple lculations. Fig. (a) shows a right-angled isosceles triangle, and g. (b), an equilateral triangle. The angles of the former are °, 45°, and 90°. The latter is divided into two triangles by opping a perpendicular from the vertex to the base as shown. he angles of the triangles so formed are 30°, 60°, and 90°.



(a)



(b)

If, in Fig. (a), we let an equal side be 1 unit, then, by t Theorem of Pythagoras, the measure of the hypotenuse $\sqrt{1^2 + 1^2} = \sqrt{2}$. Hence

$$\sin 45° = \frac{1}{\sqrt{2}}\,;\quad \cos 45° = \frac{1}{\sqrt{2}}\,;\quad \tan 45° = 1.$$

In Fig. (b), let a side of the given triangle be 2 units. Th the segments of the base are each 1 unit. If we call the altitud units, the Theorem of Pythagoras gives

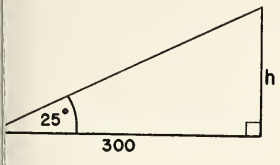
$4 = h^2 + 1$, whence $h^2 = 3$ and $h = \sqrt{3}$.

Therefore

$$\sin 30° = 1/2;\quad \cos 30° = \sqrt{3}/2;\quad \tan 30° = 1/\sqrt{3};$$
$$\sin 60° = \sqrt{3}/2;\quad \cos 60° = 1/2;\quad \tan 60° = \sqrt{3}.$$

NOTE: Figs. (a) and (b) show that the sides of a right-angl isosceles triangle are always in the ratios $1:1:\sqrt{2}$, and those of triangle with angles 30°, 60°, and 90°, are always in the rati $1:2:\sqrt{3}$.

Example 1. At a point on level ground 300 feet from the base of water-tank, an observer finds the angle of elevation of the top be 25°. How high is the tank?



The figure shows the data. Let the height of the tank be h feet.

$$\frac{h}{300} = \tan 25°.$$

Since the tables give tan 25° = 0.423, we have $h/300 = 0.423$, whence $h = 126.9$.

Therefore, the height of the tank is 130 feet, to two figures.

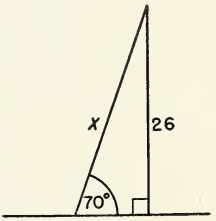
ample 2. The sill of a window is 26 ft. above the ground. How g must a ladder be to reach the sill and make an angle of 70° h the ground?

t the ladder be x ft. long. From the ure, we have

$$26/x = \sin 70° = 0.940.$$

erefore, $x = 26/0.940 = 27.7$.

e ladder should be about 28 feet long.



EXERCISES

For oral answers

e the trigonometric table at the end of the book to find the lowing ratios:

1. sin 20°.	**2.** cos 63°.	**3.** tan 17°.
4. cos 11°.	**5.** sin 79°.	**6.** tan 51°.
7. cos 85°.	**8.** tan 87°.	**9.** sin 48°.

10. Study the table of trigonometric ratios and then write in ur workbook the following sentence with the blanks filled in: the angle increases, its sine....., its cosine....., d its tangent .....

Use the table at the end of the book to find the angle A for wh

11. $\sin A = 0.326$.
12. $\cos A = 0.996$.
13. $\tan A = 1.73$.
14. $\cos A = 0.515$.
15. $\tan A = 0.577$
16. $\sin A = 0.974$.
17. $\tan A = 14.3$.
18. $\sin A = 0.766$.
19. $\cos A = 0.070$.
20. $\sin A = 0.809$.

For written answers

20. Draw with a protractor an angle of 40°. From a point on one of the arms of this angle drop a perpendicular to the oth arm. Measure the sides of the triangle so formed and compu sin 40°, cos 40°, and tan 40°, to two figures. Repeat the proc for two other positions of A and average your results. Determi the percentage error in your work by comparing your resu with those in the table.

21. Repeat the steps of Ex. 20 for an angle of 72°.

22. The angle of elevation of the top of a building is 32° a point 240 feet from the base of the building and on the sa level. How high is the building?

23. The sill of a window is 32 feet above the ground. A lad making an angle of 65° with the ground just reaches the sill. H far is the foot of the ladder from the wall?

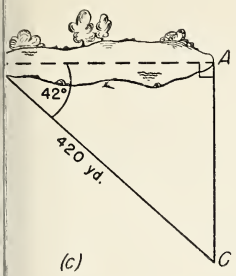
24. A man walks up a hill for a distance of 500 yards. If t hill is inclined to the horizontal at 12°, how far did the man asce vertically during his walk?

25. In Ex. 24, how far did the man travel horizonta during his walk?

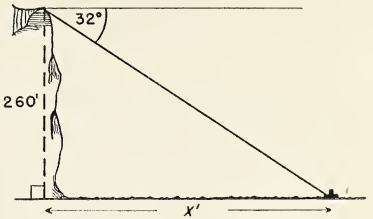
26. While trolling, a fisherman gets his line snagged bottom. He notices that the line is taut and makes an an of about 30° with the horizontal. If he has 150 feet of li out, find the approximate depth of the water where the li is caught.

27. Find the width of the lake in Fig. (c).

28. Find the distance of the buoy from the base of the c in Fig. (d).



(c)



(d)

29. A boy flies a kite with a string 250 feet long. When the ing makes an angle of 36° with the ground, find the height of kite.

30. The Eiffel tower in Paris is 980 feet high. Find the length ts shadow when the sun is 63° above the horizon.

Direct Variation

You frequently hear statements like the following:

(1) the more you buy, the more you pay;
(2) the bigger you are, the harder you fall;
(3) the faster you travel, the farther you go;
(4) the heavier the parcel, the more the postage.

ile such statements lack something in precision, each sug- ts two variables, related in such a way that the variables w or diminish together. For example, (1) suggests that the re coal you buy, the more money you pay: as the number of s increases or diminishes, the cost increases or diminishes with When two related variables grow and diminish together, they ibit ***one*** characteristic of ***direct variation.*** Other character- cs are brought to light by the next example.

Consider a ship sailing on a straight course at a uniform speed 15 m.p.h. We have the following table connecting distances ered with the times elapsed.

distance in miles, d	15	30	45	60	75	90	105
time in hours, t	1	2	3	4	5	6	7

This table shows that, although d and t are variables, the ratio their corresponding values is always constant, namely 15.

$$d/t = 15$$

This is just another way of writing the familiar formula co necting the time, rate, and distance.

If two variables x and y are so related that their ratio $x:y$ is constant, then x is said to ***vary directly as y,*** *or* ***to be proportional to y.***

Thus, *if m varies directly as n, we can write $m/n = k$, or $m =$* where k is some constant. In a given problem, it is usually p sible to find the particular value of k from the data.

Example 1. (*a*) In buying articles of a certain kind, the total c varies directly as, or is proportional to, the number of artic bought. ($c = pn$)

(*b*) In sharing partnership profits, the partners may ag that profits are to be proportional to investments. Then t ratio of profits to investments must be the same for each partn

Example 2. If x varies directly as y and $x = 10$ when $y = 2$, fi y when $x = 2.5$.

Since x varies as y, $x = ky$, where k is a constant. Sin $x = 10$ and $y = 2$ are corresponding values, we have

$$10 = k(2), \text{ whence } k = 5.$$

Therefore, the constant for this particular problem is 5 a we can write $x = 5y$. Hence, if $x = 2.5$, $y = 2.5/5 = 1/2$

RE: The words "*varies directly as*" are sometimes contracted "*varies as*". Thus when we say "*x varies as y*", we mean that *varies directly as y*".

mple 3. A merchant pays \$50.43 for £12 6s. What should paid for £18 10s?

The number of dollars paid, d, should be proportional to the number of pounds received, p. In other words, the ratio d/p is constant. From the first transaction, we find

$$d/p = 50.43/12.3.$$

Therefore, in the second transaction the ratio has the same value. This gives

$$d/18.5 = 50.43/12.3,$$

whence,
$$d = \frac{50.43 \times 18.5}{12.3} = 75.85.$$

Hence, the required price is \$75.85.

EXERCISES

For oral answers

d the following sentences with the blanks filled in:

1. If in the formula $A = lw$ the value of l is 20, then ..... es directly as ....., and the ratio $A : w$ is .....

2. If $d = 10r$, we say that d ..... r, or that d is .....

3. If x varies directly as y, the ratio $x:y$ is ..... or uals .....

4. If m is proportional to n, then m is said to ..... , the ratio $m : n$ is ....., and m equals .....

5. If $c = 30n$, then c/n is ..... and c ..... n.

For written answers

6. If x varies directly as y and $x = 5$ when $y = 2$, find the tant involved in the relation connecting x and y.

7. If a is proportional to b, and $a = 32$ when $b = 8$, (*a*) the constant involved in the relation, (*b*) the value of when $b = 6$.

8. The amount of extension in a spring balance is proportional to the weight on the hook. If 85 gm. produces an extension of 1.6 cm., what extension is produced by 136 gm.?

9. If the distance travelled by a car varies directly as the amount of gasoline consumed, and if a car goes 291.5 miles on 13 gallons, find the number of gallons required for a 450-mile trip.

10. If a merchant pays \$30.55 for £7 12s, what should he pay for £11 8s?

11. The circumference of a circle varies as the diameter. If the circumference is 154 in. when the diameter is 49 in., (*a*) the constant involved in the relation, (*b*) the circumference when the diameter is 84 in.

12. The current passing through a piece of wire is proportional to the potential difference between the ends of the wire. If the potential difference is 80 volts, the current is 4 amperes. What potential difference is required for a current of 6.2 amperes?

13. The velocity of a certain kind of wave motion varies directly as the length of the waves. It is found that waves 12 cm. long travel at 480 cm. per sec. What wave-length would produce a velocity of 1500 cm. per sec.?

14. A dealer pays \$148.75 for 17 chairs of a certain type. Use the method of direct variation to find the number of chairs he should get for \$350.

15. The volume of a certain tank is proportional to its height. If a tank 12′ 6″ high has a volume of 1300 cu. ft., how high should a tank of this type be to have a volume of 3120 cu. ft.?

Inverse Variation

If you have \$100 with which to buy hats, then you will be interested in this principle: the higher the price, the fewer hats you get. The number of dollars in the price of a hat is related to

nber of hats purchased in such a way that when the first nber increases, the second number decreases and vice versa. en two related variables behave in this way, they exhibit ? property of ***inverse variation.*** The exact nature of inverse iation is illustrated by the following example.

Consider a ship which is required to make a 360-mile voyage. ? time required for the voyage depends on the average speed. he ship travels at an average speed of r m.p.h., the numbers ours required for several values of r are given in the following le:

Number of m.p.h., r	6	9	10	12	15	18	20
Number of hours, h	60	40	36	30	24	20	18

We see that r and h are so related that as r increases, h de- ases, and the product rh is a constant, 360.

> *When two variables are so related that their product is constant, either variable is said to* ***vary inversely*** *as the other.*

refore, *if* x *varies inversely as* y,

$$xy = k, \text{ or } x = \frac{k}{y}, \text{ or } y = \frac{k}{x}.$$

This type of variation is very common. For example, if you e a fixed sum to spend on candy bars, you will find that the nber of bars you can purchase varies inversely as the price bar. As prices go up, the number of articles that can be chased for a given sum goes down.

mple. In physics we learn that the vibration frequency of a tched string varies inversely as its length. If a string 120 cm. is stretched to give a frequency of 240 vibrations per sec.,

find the frequency of a similar string 160 cm. long. (It is assu that the stretching force is the same in each case.)

Denote the number of vibrations per second by n, and number of centimetres in the length by l. Then since n va inversely as l, we have

$$nl = k, \text{ where } k \text{ is a constant.}$$

Since $n = 240$ and $l = 120$ are corresponding values, there

$$240 \times 120 = k.$$

This gives $$nl = 120 \times 240.$$

Hence, for a string 160 cm. long, we have

$$n \times 160 = 120 \times 240, \text{ whence } n = 18$$

The required frequency is 180 vibrations per second.

NOTE: Problems such as the foregoing can be easily solved observing the meaning of inverse variation. An inspection of table on page 71 shows that when r is doubled, h is hal when r is halved, h is doubled; when r is trebled, h becomes third of its former value. In other words, *when r changes certain factor, h changes by the reciprocal of that factor.* For exam in the problem just considered, the length of the strin ***increased*** by the factor 160/120 or 4/3; then the frequenc ***decreased*** by the factor 3/4, becoming $\frac{3}{4}$ of 240 or 180.

EXERCISES

For oral answers

Read the following statements with the blanks filled in:

1. If a varies inversely as b, then $ab =$ ..... $a =$ ....., and $b =$ .....

2. In the formula $A = lw$, if $A = 100$, then l ..... and w ..... l.

3. If x varies inversely as y, then as x ....., y .....

4. In the formula $C = E/R$, if $E = 110$, then C .....

5. If x varies inversely as y, then when x is doubled, ... ; when x is halved, y is ..... ; when x is tre

becomes .....; when x is increased by the factor 5/3, y ..... by the factor .....; when y is decreased the factor 3/4, x is ..... by the factor .....

For written answers

6. If x varies inversely as y and $x = 5$ when $y = 3$, find the nstant of variation.

7. If x varies inversely as y and $x = 3.5$ when $y = 2.2$, find the constant of variation, (*b*) the value y takes when $x = 1.4$.

8. If m varies inversely as n and $m = 3.2$ when $n = 5$, find value of m when $n = 6.4$.

9. If the area of a rectangle is constant, the length varies ersely as the width. Express this fact in the symbols of ebra. If the length is 8 in. when the width is 12 in., find the nstant value of the area. What width corresponds to a length 16 in.?

10. If a varies inversely as b and $a = 60$ when $b = 100$, write sight (*a*) the value of b when a is doubled, (*b*) the value of a en b is quartered, (*c*) the value of b when a is increased by the tor 4/3, (*d*) the value of a when b is increased by the factor 6/5.

11. The frequency of vibration of a stretched string varies ersely as its length, other things being equal. If a string 80 cm. g vibrates 180 times every second, find (*a*) the frequency when string is shortened to 60 cm., (*b*) the length of the string uired for a frequency of 240 vibrations per second.

12. The frequency of a stretched string varies inversely as its meter, other things being equal. If a string with diameter 5 cm. vibrates 100 times per second, find (*a*) the frequency of imilar string with diameter 0.03 cm., (*b*) the diameter of a ilar string with a frequency of 40 vibrations per second.

13. In a certain electric circuit, the current flow is inversely portional to the resistance (i.e., varies inversely as the re- ance). When the resistance is 4 ohms, the current is amperes. Find (*a*) the current when the resistance is 4.8 ns, (*b*) the resistance when the current is 38.4 amperes.

14. If the pressure of a given mass of gas varies inversely its volume, find the constant of variation, given that the volu is 1000 cu. ft. when the pressure is 14 lb. per square in Find the volume of this gas when the pressure is (*a*) 3.5 lb. sq. in., (*b*) 52.5 lb. per sq. in.

REVIEW EXERCISES ON CHAPTER III

A

1. What is meant by *the ratio of two numbers*?

2. Represent, in two different ways, two general numbers the ratio 2:3.

3. If two numbers are in the ratio 4:7, write two facts t are known about the numbers.

4. Two numbers are in the ratio of 4:7 and their sum is Find the numbers.

5. Divide 340 into two parts in the ratio 8:9.

6. What is meant by *the ratio of two quantities of the same ki*

7. Divide \$315 into three parts in the ratios 5:7:9.

8. Find the ratio of (*a*) 8 feet to 2 yards, (*b*) 200 minutes 4 hours, (*c*) 60 miles per hour to 44 feet per second, (*d*) 3.5 to 28 oz.

9. If 3, 5, 6, x are in proportion, find x.

10. Solve $7/2 = 14/x$ for x.

11. If $x : 11 = 12 : 33$, find x.

12. What is the value of $\sin 30° \cdot \cos 60° \cdot \tan 45°$?

13. Draw with a protractor an angle of 35°. From a poin on one arm of this angle drop a perpendicular to the other a Measure the sides of the triangle so formed and find, to 2 figu sin 35°, cos 35°, and tan 35°.

14. What is meant by the statement (*a*) *x varies directly as* (*b*) *x varies inversely as y*?

15. If x varies directly as y and $x = 20$ when $y = 4$, find constant of variation.

16. If r varies inversely as t and $r = 2.5$ when $t = 4$, find constant of variation.

B

7. What number must be added to each term of the ratio 17 to produce a ratio equal to 5:7?

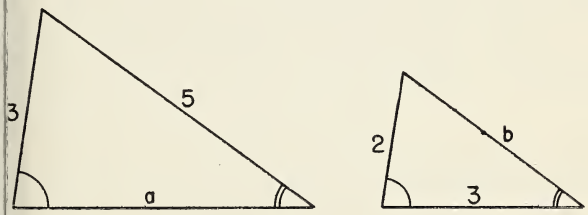
8. A 720-foot chain is divided into three lengths in the os 5 : 6 : 7. Find the length of the shortest piece of chain.

9. Find a mean proportional between (*a*) 9 and 25, (*b*) x^2y x^4y^3.

0. What number must be added to each of the numbers 0, 7, 25 so that the sums will be in proportion?

1. What number must be subtracted from each of the ıbers 15, 36, 13, 31 so that the differences will be in pro- tion?

2. Find the values of a and b for the following similar ngles:



3. The shadow of a vertically held 6-foot stick was 3′ 6″ ;, and at the same time the shadow of a tree was 42 feet ;. How high was the tree?

4. The statue of a man 6 feet tall is 15 feet in height. (*a*) What ıe ratio of the height of the statue to the height of the man? If the man's nose is 2.4 inches long, what is the length of the e on the statue? (*c*) If the index finger on the left hand of the ue is 9 inches long, what is the length of the corresponding er of the man?

5. From a point 160 feet from its base, the angle of elevation he top of a monument is 58°. Find the height of the ıument.

26. A wire is stretched from the top of a telephone pole t point on the ground 24 feet from its base. If the wire makes angle of 62° with the ground, find the length of the wire.

27. A ladder 40 feet long has its upper end on the sill o window, and makes an angle of 50° with the ground at its lo end. Find the height of the window sill above ground level

28. If x varies directly as y and $x = 4.5$ when $y = 7.5$, f the value of y when $x = 9.6$.

29. The speed of a ship in miles per hour varies directly as speed in knots. If a speed of 38 m.p.h. is equal to a speed 33 knots, express 11 knots in miles per hour.

30. If A varies inversely as B and $A = 4\frac{3}{4}$ when $B =$ find B when $A = 7\frac{1}{2}$.

31. If the area of a triangle is constant, the base varies versely as the height. Express this fact in the symbols of algeb If the height is 12.5 units when the base is 18.6 units, find constant of variation. What height corresponds to a base 31 units?

C

32. When a tree broke off in a storm, the top of the tree str the ground 56 feet from the trunk. The line from the break the tree to the point where the top struck the ground was served to make 38° with the ground. Find the height of the before it fell.

33. A ladder 32 feet long has its upper end at the top vertical wall and its lower end on level ground 15 feet from base of the wall. Find (*a*) the angle between the ladder and ground, (*b*) the height of the wall.

34. The area of a square inscribed in a circle varies dire as the area of the circle. When the area of the circle is 15.7, area of the square is 10. Find the area of the square when area of the circle is 20.

35. Translate the following statements into the symbol algebra: (*a*) x varies directly as the square of y; (*b*) z va

ctly as the cube of r; (c) m varies inversely as the square of n; u varies inversely as the cube of b.

. The radii of three circles have the ratios 1:2:3. What are ratios of their areas?

. The prices of two articles are in the ratio 5:8. If the total of three of the first and 10 of the second is \$43.70, find the es of the articles.

. Find two numbers whose sum is 34 and whose mean pro- ional is 15.

. A rectangle and a square have the same perimeter, and areas are in the ratio 24:25. Find the ratio of the lengths ne sides of the rectangle.

. If $\frac{x}{a} = \frac{y}{b} = \frac{5}{8}$, find the value of $\frac{6x - 7y}{6a - 7b}$.

. John's age is to Tom's age as 5:6. Three years ago the was 4:5. Find their ages.

. When a certain sum of money is divided in the ratio 4:5 smaller portion is \$2 more than when the money is divided ne ratio 5:7. Find the sum of money.

CHAPTER IV

FACTORING

The Meaning of Factoring

Factoring in arithmetic is usually confined to whole numb
We say that 4 and 3 are factors of 12 since $4 \times 3 = 12$; simila
2, 2, 3, and 5 are factors of 60 since $2 \times 2 \times 3 \times 5 = 60$.
is a whole number, then any whole number which divides in
without remainder is a factor of n. A whole number like 7 w
has no factors except itself and 1 is called a ***prime num***

In algebra, the idea of factor is somewhat more general. T
the factors of the monomial $3ab$ are 3, a, and b, regardles
whether a and b stand for whole numbers or not. In the s
way, 3, 2, x, x, and y are factors of $6x^2y$ for any values of x an

Much of the factoring in this course is concerned with
pressions like

$$8x^2 - 24x \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$4x^2 - 9y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$a^2b^2 - 5ab + 6 \quad . \quad . \quad . \quad . \quad . \quad (3)$$
$$27x^4y^3 - 3x^3y^3 + 9x^2y^3 \quad . \quad . \quad . \quad (4)$$

Such expressions are called ***polynomials***. Their terms co
of either one or both of the following:

(*a*) numerical constants,

(*b*) literal numbers which are products of simple le
numbers like a, b, x, y with numerical coefficients.

Each term of the polynomial has a ***degree*** which is
number of simple letter-factors in its literal part; e.g., s
$x^2y^3 = xxyyy$, the term $6x^2y^3$ is of degree 5; similarly, $-8x^3y$
degree 4 and $\frac{3}{4}xy^2z^3w^3$ is of degree 9.

> *If, in a polynomial, there is a term of degree k and no term of higher degree, then k is called* ***the degree of the polynomial.***

e numerical constants such as 11, $-4/5$, 8.3, involve no rs, they are considered to be polynomials of degree 0. *If all erms of a polynomial have the same degree, the polynomial is to be* ***homogeneous.***

Hence, of the foregoing polynomials, (1) is of degree 2, (2) of ee 2 and homogeneous, (3) of degree 4, and (4) of degree 7. Consider now the process of factoring these polynomials. have

$$8x^2 - 24x = 8x(x - 3),$$
$$4x^2 - 9y^2 = (2x + 3y)(2x - 3y),$$
$$a^2b^2 - 5ab + 6 = (ab - 3)(ab - 2),$$
$$27x^4y^3 - 3x^3y^3 + 9x^2y^3 = 3x^2y^3(9x^2 - x + 3).$$

ach case, the polynomial has been written as the product of nomials. By examining the degrees of the expressions on sides of the equalities, and by extending this examination any other examples from our experience, we are led to accept following definition:

> ***To factor a polynomial is to express it as the product of polynomials of lower degrees.***

Uses of Factoring

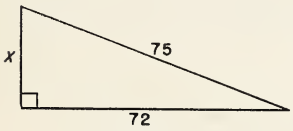
You have already learned some methods of factoring. In the ent chapter, new methods and more complicated examples be introduced. At this point it is proper to ask: What is the pose of factoring? The answer is sometimes given that the pring of expressions gives valuable training in recognizing portant algebraic forms. Such training is certainly useful for e who proceed to more advanced mathematics. However, pring has some uses that may be appreciated more readily.

One of the most common manipulations in many branches of hematics, including algebra, is that of ***transforming an ression*** from one form to another. This transformation may arried out for any one of several reasons. The transformation

may yield a much simpler expression, or one more conve
for a certain operation, or one from which interesting and u
conclusions may be easily drawn, or one better suited to
stitution or computation. Skill in transforming expressio
helpful and often necessary in the work of this course as w
that of future courses in mathematics.

*Factoring is one of our most effective methods for the t
formation of expressions.* The following examples illustrate
of the ways in which factoring may be used.

Example 1. A common problem of geometry and trigonome
that of finding the third side of a right triangle when th
potenuse and one side are given. Suppose we have the
marked on the figure.



If we let the third side be x u
then the Theorem of Pytha
gives

$$x^2 = 75^2 - 72^2$$
$$= (75 + 72)(75 - 72)$$
$$= 147 \times 3 = 441.$$

Therefore $x = 21$, and the third side is 21 units. In
example we made the computation less laborious by fact
the difference of two squares.

Example 2. Solve: $2x^2 + x - 6 = 0$.

Factoring the left side, we have

$$(2x - 3)(x + 2) = 0.$$

Therefore, $2x - 3 = 0$ or $x + 2 = 0$,

whence $x = 3/2$ or -2.

Here the factoring enabled us to transform the left
of the equation into a form from which a conclusion
readily be drawn. This method will be further devel
in Chapter VII.

nple 3. Many formulas are given a more convenient form actoring. The "s" formula on page 319 is an example. Here ome other examples.

a) The surface of a cylindrical tin can is given by

$$A = 2\pi R^2 + 2\pi Rh = 2\pi R(R + h).$$

b) The surface of a flat circular ring is given by

$$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(R + r)\,(R - r).$$

nple 4. Many operations with fractions are effected with the of factoring. Consider the following.

$$\frac{x^2 - y^2}{x^3 + x^2y} = \frac{\overset{1}{\cancel{(x + y)}}(x - y)}{x^2\underset{1}{\cancel{(x + y)}}} = \frac{x - y}{x^2};$$

$$\left(\frac{3a - 6}{a^2 - 5a + 6}\right)\left(\frac{a^2 - 9}{3a + 9}\right) = \frac{\overset{1}{\cancel{3}}\overset{1}{\cancel{(a - 2)}}\overset{1}{\cancel{(a + 3)}}\overset{1}{\cancel{(a - 3)}}}{\underset{1}{\cancel{(a - 3)}}\underset{1}{\cancel{(a - 2)}}\underset{1}{\cancel{3}}\underset{1}{\cancel{(a + 3)}}} = 1;$$

$$\frac{1}{x + y} + \frac{y}{x^2 - y^2} = \frac{(x - y) + y}{(x + y)(x - y)} = \frac{x}{x^2 - y^2}.$$

n order to transform many other expressions, we shall find new methods of factoring helpful. Before proceeding to types, let us first recall those considered so far.

Review of Factoring

t will be sufficient for us to comment briefly on some methods examine some illustrative examples.

actors common to every term. When all the terms of a nomial have a factor in common, we can factor the poly- al as follows:

(1) Write the common factor; then divide it mentally i
the polynomial.
(2) Place the quotient in brackets beside the common fac

Example 1. $3x^2 - 15xy + 12xy^2 = 3x(x - 5y + 4y^2)$.

Example 2. $(a + b)x - (a + b)y = (a + b)(x - y)$.

Trinomials. Polynomials with three terms such as

$$3x^2 + 4x - 15$$

are sometimes called trinomials. They may often be factored
inspection. For instance, in the example just noted, the term
suggests the factors $3x$ and x, and 15 suggests 5 times 3 o
times 1. We have the following possibilities:

$$\begin{matrix} 3x & 3 & 3x & 5 & 3x & 1 & 3x \\ x & 5 & x & 3 & x & 15 & x \end{matrix}$$

By inspection we see that the second possibility is the only
that can yield a product term $4x$. The proper signs are
inserted and the resulting factors checked mentally by
tiplication. We get

$$3x^2 + 4x - 15 = (3x - 5)(x + 3).$$

Example 3. $5x^2 - 26x + 5 = (5x - 1)(x - 5)$.

Example 4. Factor $(2x^2 - 5x)^2 - (2x^2 - 5x) - 6$.

This expression takes the form of a trinomial if we re
$2x^2 - 5x$ as one number. Let $2x^2 - 5x = M$, and we ha

$$M^2 - M - 6 = (M - 3)(M + 2).$$

Replacing M by $2x^2 - 5x$, we get the factors

$$(2x^2 - 5x - 3)(2x^2 - 5x + 2)$$
$$= (2x + 1)(x - 3)(2x - 1)(x - 2).$$

The ability to obtain factors by this method of shrewd g
ing is improved with practice. Later, in the study of quadr
we shall find a formula for factoring more difficult trinomi

NOTE. If the trinomial $16x^2 - 40xy + 25y^2$ is factored by
we get two equal factors:

$$(4x - 5y)(4x - 5y).$$

means that the given trinomial is a perfect square, $(4x - 5y)^2$. same result could have been obtained by recalling the ex- sion of $(a - b)^2$.

actors of the difference of two squares. Whenever a poly- ial has or can be made to have the form

$$(\quad)^2 - (\quad)^2,$$

an factor it by comparison with $x^2 - y^2$. The equality

$$\boxed{x^2 - y^2 = (x + y)(x - y)}$$

formula for factoring the difference of the squares of two bers. It is instructive to translate the formula into English:

The difference of the squares of two numbers equals the sum of the numbers times their difference.

nple 5. $9x^2 - 4y^2 = (3x)^2 - (2y)^2$
$= (3x + 2y)(3x - 2y).$

E: With practice the first step in this solution becomes cessary.

nple 6. $25a^2 - (b + 2c)^2 = (5a + \overline{b + 2c})(5a - \overline{b + 2c})$
$= (5a + b + 2c)(5a - b - 2c).$

he foregoing formula for factoring the difference of two res is sometimes useful in finding products. (See Chap. I, 5.) The formula may be translated into English thus:

The sum of two numbers times their difference equals the square of the first number less the square of the second.

: We meet here an important contrast between $a^2 - b^2$ and $a^2 + b^2$. former, $a^2 - b^2$, has two linear factors while the latter, $a^2 + b^2$, has no factors.

nple 7. $102 \times 98 = (100 + 2)(100 - 2)$
$= 100^2 - 2^2 = 9996.$

Example 8. $(a + b - c)(a - b + c) = (a + \overline{b - c})(a - \overline{b}$
$= a^2 - (b - c)^2$
$= a^2 - (b^2 + c^2 - 2bc$
$= a^2 - b^2 - c^2 + 2bc.$

EXERCISES

For oral answers

1. What is meant by factoring (*a*) whole numbers? (*b*) p nomials?

2. Complete the following statements: (*a*) The differen the squares of two numbers equals .....; (*b*) the su two numbers times their difference equals .....

3. Evaluate: (*a*) $52^2 - 48^2$; (*b*) $33^2 - 27^2$; (*c*) $104^2 - 9$

4. Two squares have sides 91 inches and 9 inches, res tively. Find the difference in their areas.

5. Two squares have sides 45 cm. and 35 cm., respecti Find the difference in their areas.

Factor the following:

6. $4x - 8x^2$.	7. $ax^2 + ay^2$.	8. $x^2 + 8x +$
9. $r^2 - s^2$.	10. $4x^2 - 12x$.	11. $a^2 - 9$.
12. $a^2 - 3a - 40$.	13. $x^2 - 7x + 12$.	14. $4x^2 - 1$.
15. $10x^2y^2 + 15xy$.	16. $x^2 - x - 20$.	17. $2x^2 + 3x +$
18. $cd^3 + cd^2$.	19. $1 - 25n^2$.	20. $1 - 8x + 1$
	21. $9 - 12x + 4x^2$.	

For written answers

Use the method of Examples 7 and 8 to obtain the follo products:

21. $(a + b + c)(a + b - c)$.
22. 103×97.
23. $(x - 2y + 3z)(x - 2y - 3z)$.
24. $(m + n - 2k)(m - n + 2k)$.
25. $(3x - y - z)(3x + y + z)$.
26. 65×55.

7. $(r + s + 2t)\,(r + s - 2t)$.
8. $(5x - y - z)\,(5x + y + z)$.
9. $(b + c + d)\,(-b + c + d)$.

tor as completely as possible:

0. $2x^2 + x - 1$.
2. $z^3 - z$.
4. $ax^2 + 5ax + 6a$.
6. $3y^4 - 12y^2$.
8. $4x^2 - 32x + 60$.
0. $x^2 - \frac{1}{4}$.
2. $9x^2y^2 - 1$.
4. $3x^2 + 17x + 10$.
6. $8x^2 + x - 7$.
8. $36ab - 12a^2 - 24ac$.
0. $81 - y^4$.
2. $2a^2 - 22a - 52$.
4. $6x^2 - 3x - 30$.
6. $12x^2 + 107xy - 9y^2$.
8. $12x^2 + 52xy - 9y^2$.
0. $(m + n)x + (m + n)y$.
2. $(c + d)^2 - n^2$.

31. $2x^2 - 8$.
33. $2b^3 - 4b^2 - 6b$.
35. $x^4 - x^2$.
37. $mx^3 - mx^2$.
39. $3m^2 - 2m - 1$.
41. $n^2 - 12n + 36$.
43. $10x^2 + 15x - 10$.
45. $12a^2 + 5a - 3$.
47. $1 - 4k^2m^2$.
49. $x^4 - 1$.
51. $x^3y + 2x^2y + xy$.
53. $6x^2 + 5x - 4$.
55. $km^2 - 25k$.
57. $12x^2 - 31xy + 9y^2$.
59. $25a^2 - 40ab + 16b^2$.
61. $(m + n)x^2 - (m + n)y^2$.
63. $k^2 - (m + n)^2$.

4. $(p + q)a + (p + q)b - (p + q)$.
5. $(2x - 3y)^2 - 25$.
6. $(x^2 + 5x)^2 + 11(x^2 + 5x) + 30$.
7. $(3x^2 - 2x)^2 - 9(3x^2 - 2x) + 8$.
8. $(2x - 5y)x + 3(2x - 5y)y$.
9. $(a + b)^2 - (c - d)^2$.
1. $(x - y)^2 - 5(x - y)$.
3. $6x^2 - 25xy + 24y^2$.

70. $15a^2 - 14ab - 8b^2$.
72. $(m + n)^4 - (m + n)^2$.

4. Find, by factoring, the quotient when the product of $^2 - 29x + 10$ and $3x^2 + 13x + 12$ is divided by $- 7x - 20$.

5. Find, by factoring, the quotient when the product of $^2 - 9b^2$ and $14a^2 - 53ab + 14b^2$ is divided by $7a(4a + 3b)$ $2b(4a + 3b)$.

76. Simplify each of the following by factoring:
(*a*) $(x^2 - 7x + 2)^2 - (x^2 - 7x + 1)^2$,
(*b*) $(5a - 2b + 3c)^2 - (5a + 2b - 3c)^2$.

Factoring by Grouping

Sometimes by inserting brackets we can group the terms of an expression so as to reduce it to a form which is more readily factorable.

Grouping to get a common factor. Consider the expression

$$ax - ay + bx - by \quad . \quad . \quad . \quad . \quad (1)$$

There is no factor common to the four terms of (1). However, by grouping the first two terms and the last two, we can change the expression into one with two terms, each of which has the factor $x - y$. We have

$$\begin{aligned}(ax - ay) + (bx - by) &= a(x - y) + b(x - y)\\ &= (x - y)\,(a + b).\end{aligned}$$

Alternatively, we might have grouped the first and third terms together and the second and fourth together, thus:

$$\begin{aligned}(ax + bx) - (ay + by) &= x(a + b) - y(a + b)\\ &= (a + b)\,(x - y).\end{aligned}$$

Recall that *when terms are placed within parentheses with a minus sign in front, the signs of the terms must be changed.*

Example 1.

$$\begin{aligned}3x^3 + 6x^2 - 2x - 4 &= (3x^3 + 6x^2) - (2x + 4)\\ &= 3x^2(x + 2) - 2(x + 2)\\ &= (x + 2)\,(3x^2 - 2).\end{aligned}$$

Try the alternative grouping $(3x^3 - 2x) + (6x^2 - 4)$.

Example 2.

$$\begin{aligned}m^2 - 4n^2 - 7m + 14n &= (m^2 - 4n^2) - (7m - 14n)\\ &= (m + 2n)\,(m - 2n) - 7(m -\\ &= (m - 2n)\,(m + 2n - 7).\end{aligned}$$

Grouping to get the difference of squares. If, in an expres-
ı, there exist terms which combine to form a perfect square,
ıay be possible to group terms and obtain the form

$$(\quad)^2 - (\quad)^2.$$

ımple 3. $a^2 + 2ab + b^2 - c^2 = (a^2 + 2ab + b^2) - c^2$
$= (a + b)^2 - c^2$
$= (a + b + c)\,(a + b - c).$

ımple 4. $2mn - m^2 - n^2 + 1 = 1 - (m^2 + n^2 - 2mn)$
$= 1^2 - (m - n)^2$
$= (1 + \overline{m - n})\,(1 - \overline{m - n})$
$= (1 + m - n)\,(1 - m + n).$

ɛ: A common error is that of multiplying an ***expression*** by some
ıber in order to eliminate fractions or minus signs. For example, you
ht be tempted to replace the expression in Example 4 by $m^2 + n^2 -$
$- 1$. This would lead to an incorrect answer; the two expressions are
equivalent. ***If you multiply an expression other than 0 by any
nber except 1, you change its value.*** Thus the value of $-x - 1$ is
ıged if we multiply it by -1.

Now notice the contrast with the ***equation*** $-x - 1 = 0$. One of our
c principles allows us to multiply ***both*** sides of the equation by -1 to
the equivalent equation $x + 1 = 0$.

ımple 5. Factor $a^2 + 4b^2 - 9c^2 - 25d^2 - 4ab + 30cd$.
Grouping the terms which combine to give squares, we have

$$(a^2-4ab+4b^2)-(9c^2-30cd+25d^2)=(a-2b)^2-(3c-5d)^2$$
$$=(a-2b+3c-5d)(a-2b-3c+5d).$$

EXERCISES

tor each of the following by grouping so as to get a common
ɔr:

1. $cx + cy + dx + dy.$
2. $4k - 4m + ak - am.$
3. $px + py - qx - qy.$
4. $am - an - bm + bn.$
5. $3x^3 - 6x^2 - 5x + 10.$
6. $x^3 - x^2 - x + 1.$
7. $ky - my + km - y^2.$
8. $7xy - 3xz - 7wy + 3wz.$
9. $(x + 1)^2 - x - 1.$
10. $ab - 1 - a + b.$

Factor each of the following by grouping terms so as to get difference of squares:

11. $x^2 + 2xy + y^2 - z^2$.
12. $4m^2 - 4mn + n^2 - 9k^2$
13. $a^2 - b^2 - c^2 - 2bc$.
14. $25p^2 - q^2 - 9 + 6q$.
15. $2xy - x^2 - y^2 + 1$.
16. $16x^2 + 30yz - 9y^2 - 2$
17. $8m + 36n^2 - m^2 - 16$.
18. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd$.
19. $4x^2 - 12x - y^2 + 6ay - 9a^2 + 9$.
20. $2p + 8q + 1 + p^2 - q^2 - 16$.

Factor as fully as possible:

21. $a^2 - 25b^2 - 3a - 15b$.
22. $p(x - 2y) + q(x - 2y) - x + 2y$.
23. $x^2 - 2xy + y^2 + x - y$.
24. $(2k + n)^2 - 5(2k + n) + 6k + 3n$.
25. $9a^2 - 4b^2 + 45a - 30b$.
26. $x^3 + 3x^2 - x - 3$.
27. $(5x - y)^2 - a(5x - y) - 5x + y$.
28. $5m^2 - 10mn + 5n^2 - 45k^2$.
29. $4p^2x^2 - q^2x^2 - 36p^2y^2 + 9q^2y^2$.
30. $7(a - b) + ab - b^2$.

Incomplete Squares

Consider the following expansions of complete squares:

$$(2a + b)^2 = 4a^2 + 4ab + b^2,$$
$$(m - 5k)^2 = m^2 - 10km + 25k^2,$$
$$(x^2 - 3y^2)^2 = x^4 - 6x^2y^2 + 9y^4,$$
$$(3c^2 + 5d^2)^2 = 9c^4 + 30c^2d^2 + 25d^4.$$

By inspecting such expansions, we are led to infer that expans which are squares of binomials contain

(1) two terms which are perfect squares, and
(2) the correct product term.

A polynomial like

$$4x^4 - 5x^2y^2 + y^4 \quad . \quad . \quad . \quad (1)$$

suggests the perfect squares

$$(2x^2 - y^2)^2 \quad \text{and} \quad (2x^2 + y^2)^2,$$

:pt for the product term $-5x^2y^2$. If the product term were c^2y^2 or $4x^2y^2$, then (1) would be a complete square.

To form the square $(2x^2 - y^2)^2$, we can write (1) thus:

$$4x^4 - 4x^2y^2 + y^4 - x^2y^2 \quad . \quad . \quad . \quad (2).$$

)ression (2) is equivalent to (1) and its first three terms can rouped to give the required square. Moreover, if we proceed ı this grouping, we see that (1) takes the form of the difnce of two squares and can be readily factored. We have

$$\begin{aligned} - 4x^2y^2 + y^4) - x^2y^2 &= (2x^2 - y^2)^2 - (xy)^2 \\ &= (2x^2 - y^2 + xy)\,(2x^2 - y^2 - xy) \\ &= (2x - y)\,(x + y)\,(2x + y)\,(x - y). \end{aligned}$$

Similarly, if we construct the square $(2x^2 + y^2)^2$, (1) becomes

$$\begin{aligned} + 4x^2y^2 + y^4) - 9x^2y^2 &= (2x^2 + y^2)^2 - (3xy)^2 \\ &= (2x^2 + y^2 + 3xy)\,(2x^2 + y^2 - 3xy) \\ &= (2x + y)\,(x + y)\,(2x - y)\,(x - y). \end{aligned}$$

:ach case, the completion of the square and the proper adment to restore the product term produced the difference of squares and provided a method of factoring the given inplete square. In examples such as the foregoing, it is always ible to complete the square in two different ways. However, of these methods may fail to lead to factors. For example,

$$\begin{aligned} 4a^4 + 1 &= 4a^4 - 4a^2 + 1 + 4a^2 \\ &= (2a^2 - 1)^2 + 4a^2, \end{aligned}$$

:h is the ***sum*** of squares. Try to factor this polynomial by structing the square $(2a^2 + 1)^2$.

E: (1) Polynomials which admit this type of factoring have degrees of four or more.

(2) The product term which completes the square should always be corrected by ***subtracting*** a like term. If the correction requires addition, you should complete the other possible square.

(3) Sometimes expressions of this type can also be tored as trinomials. For example,

$$4x^4 - 5x^2y^2 + y^4 = (4x^2 - y^2)(x^2 - y^2)$$
$$= (2x + y)(2x - y)(x + y)(x - y).$$

Example 1. $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2$
$= (x^2 + y^2)^2 - (xy)^2$
$= (x^2 + y^2 + xy)(x^2 + y^2 - xy)$

Example 2.
$16m^4 - 44m^2n^2 + 25n^4 = 16m^4 - 40m^2n^2 + 25n^4 - 4m^2n^2$
$= (4m^2 - 5n^2)^2 - (2mn)^2$
$= (4m^2 - 5n^2 + 2mn)(4m^2 - 5n^2 - 2$

EXERCISES

Factor:

1. $m^4 + m^2n^2 + n^4$.
2. $p^4 - 5p^2 + 4$.
3. $x^4 + 3x^2 + 4$.
4. $y^4 - 10y^2 + 9$.
5. $4m^4 + 1$.
6. $4c^4 + 3c^2 + 1$.
7. $16x^4 + 36x^2y^2 + 25y^4$.
8. $49k^4 - 51k^2m^2 + 9m^4$.
9. $a^4 + b^4 - 18a^2b^2$.
10. $a^4 + b^4 - 14a^2b^2$.
11. $25x^4 - 89x^2y^2 + 64y^4$.
12. $25x^4 + 44x^2y^2 + 64y^4$.
13. $100p^4 + 25$.
14. $m^8 + m^4 + 1$.
15. $m^6 + m^4 + m^2$.
16. $100n^4 - 29n^2 + 1$.

The Sum and the Difference of Cubes

If we divide the expression $x^3 - y^3$ by $x - y$, we get the quotient $x^2 + xy + y^2$ as shown. Similarly, if we divide $x^3 + y^3$ by $x + y$, we get the quotient $x^2 - xy + y^2$.

$$\begin{array}{r|l} & x^2+xy+y^2 \\ \hline x-y & x^3-y^3 \\ & x^3-x^2y \\ \hline & x^2y \\ & x^2y-xy^2 \\ \hline & xy^2- \\ & xy^2- \end{array}$$

These results should be memorized:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

means of these formulas, we can factor expressions of the forms

$$(\quad)^3 \pm (\quad)^3.$$

s seen that *the sum of the cubes of two numbers is divisible by r sum, and the difference of the cubes of two numbers is divisible heir difference.* The middle term in the second factor of each nula should be noted carefully:

(1) the coefficient of xy is ***not*** ± 2, as in the expansions of $(a \pm b)^2$.

(2) the sign preceding xy is ***opposite*** to that between the given cubes.

mple 1.

$$8x^3 - 125y^3 = (2x)^3 - (5y)^3$$
$$= (2x - 5y)(4x^2 + 10xy + 25y^2).$$

mple 2.

$$+ b)^3 + 27c^3 = (a + b + 3c)(\overline{a + b}^2 - 3c \cdot \overline{a + b} + 9c^2)$$
$$= (a + b + 3c)(a^2 + 2ab + b^2 - 3ac - 3bc + 9c^2).$$

EXERCISES

For oral answers

1. Translate each of the formulas given above into a rule English.

2. Give one factor of each of the following polynomials: $a^3 - b^3$, (*b*) $m^3 - 8$, (*c*) $125 + c^3$, (*d*) $8k^3 - 27$, $(x + y)^3 + z^3$.

tor:

3. $a^3 + b^3$.
4. $m^3 + 8$.
5. $x^3 - 8$.
6. $1 - 64c^3$.
7. $1 + d^3$.
8. $27x^3 - 1$.
9. $8 - n^3$.
10. $m^3 - 1$.
11. $p^3 + q^3$.

For written answers

Factor:

12. $8a^3 - b^3$. **13.** $x^3 + 64y^3$. **14.** $125m^3 + 8n^3$.

15. $216p^3 - 1000q^3$. **16.** $1 - 8c^3d^3$. **17.** $k^6 - m^3$.

18. $a^6 + b^6$. **19.** $27x^6 - 125$. **20.** $1000a^3b^3 + 1$.

21. $2x^3 + 54$. **22.** $3x^4 - 192xy^3$. **23.** $d^2 + d^5$.

24. $(a - b)^3 - c^3$. **25.** $(c - 3d)^3 + 27d^3$.

26. $16m^4n + 2mn^4$. **27.** $(3x + 5)^3 - 125$.

28. $(x + y)^3 - (x - y)^3$. **29.** $x^{12} - y^{12}$.

30. Find one factor of $(3x - 5y + 9z)^3 - (2x - 6y + 8z)^3$

31. Use factoring to show that

$$(5a - 7b - 13c)^3 + (13c + 7b - 5a)^3 = 0.$$

The Factor Theorem

When a polynomial does not readily lend itself to any of methods of factoring so far discussed, we can sometimes f factors by using the following principle:

> *If one factor of a polynomial becomes* 0, *the value of the polynomial becomes* 0.

For example, since $x^2 - x - 2 = (x + 1)(x - 2)$, the p nomial $x^2 - x - 2$ becomes 0 if either $x + 1 = 0$ or $x - 2 =$ Hence, if we make $x + 1 = 0$ by substituting -1 for x, we g

$$x^2 - x - 2 = (-1)^2 - (-1) - 2 = 1 + 1 - 2 = 0.$$

Similarly, if we set $x = 2$, we make the factor $x - 2 = 0$

$$x^2 - x - 2 = 4 - 2 - 2 = 0.$$

This principle will be more fully discussed in Chapter V

The foregoing suggests a method for finding the factors polynomial. If a polynomial in x equals 0 when we set x then one of its factors must have become 0. This means $x - 3$ is a factor of the polynomial. In the same way,

nomial in x vanishes when we set x equal $\frac{3}{4}$ (or y, or -5) $4x - 3$ (or $x - y$, or $x + 5$, respectively) is a factor of the nomial. The following theorem can be proved.

> ***The Factor Theorem***
>
> *If a polynomial in x vanishes when $x = a$, then $x - a$ is a factor of the polynomial.*

mple 1. Factor: $x^3 - 4x^2 + x + 6$.

Since this polynomial has four terms, we might first try grouping. It is easily shown that the usual methods of grouping fail to disclose a factor.

The Factor Theorem can be used as follows. Since the constant term is 6, it is reasonable to expect factors, if any, ending in 1, 2, 3, or 6. Therefore, let us substitute in succession for x as many of the values $1, -1, 2, -2, 3, -3, 6, -6$, as necessary.

If $x = 1$, the polynomial becomes
$$1 - 4 + 1 + 6 \text{ which } \neq 0.$$
If $x = -1$, the polynomial becomes
$$-1 - 4 - 1 + 6 \text{ which } = 0.$$
Hence $x + 1$ is a factor.
If $x = 2$, the polynomial becomes
$$8 - 16 + 2 + 6 \text{ which } = 0.$$
Hence $x - 2$ is a factor, and the third factor is likely to be $x - 3$ or $x + 3$ in order to give the constant term 6.
If $x = 3$, the polynomial becomes
$$27 - 36 + 3 + 6 \text{ which } = 0.$$
Hence $x - 3$ is a factor and the polynomial is
$$(x + 1)(x - 2)(x - 3),$$
which can be verified by multiplication.

It is sometimes simpler to find ***one*** factor by using the tor Theorem, and then use long division to get the other.

Example 2. Factor: $2x^3 - 9x^2 - 20x + 12$.

Let us locate a factor using the Factor Theorem.

If $x = 1$, the polynomial equals
$$2 - 9 - 20 + 12 \ (\neq 0).$$
If $x = -1$, the polynomial equals
$$-2 - 9 + 20 + 12 \ (\neq 0).$$
If $x = 2$, the polynomial equals
$$16 - 36 - 40 + 12 \ (\neq 0).$$
If $x = -2$, the polynomial equals
$$-16 - 36 + 40 + 12 \ (= 0).$$

Therefore $x + 2$ is a factor. By long division, we find
$$2x^3 - 9x^2 - 20x + 12 = (x + 2)(2x^2 - 13x + 6)$$
$$= (x + 2)(2x - 1)(x - 6).$$

EXERCISES

1. Show that $x - 1$ is a factor of $x^3 - 5x^2 - x + 5$.
2. Find which one of $x - 2$, $x + 3$, $x - 5$, is a factor of $x^3 + x^2 - 5x - 2$.
3. Find one factor of $x^3 - 10x + 3$.
4. Show that $x + 5$ is a factor of $2x^3 + 10x^2 - 3x - 15$.
5. Show that $2x - 1$ is a factor of $8x^3 + 2x^2 - x - 1$.

Factor the next four polynomials given that each is divisible by $x + 1$, $x - 1$, $x + 2$, or $x - 2$.

6. $x^3 - 2x^2 + x - 2$.
7. $x^3 + 7x^2 - x - 7$.
8. $3x^3 + 4x^2 - 5x - 2$.
9. $2x^3 + x^2 - 5x - 4$.
10. Show that $x - 2$ is a factor of $x^3 - 4x^2 + x + 6$, and find the other factors.
11. Show that $a + 1$ is a factor of $2a^3 + 3a^2 - 9a - 10$, and find the other factors.

Factor fully:

12. $4x^3 + 12x^2 - x - 3$.
13. $x^3 - 5x^2y - xy^2 + 5y^3$.
14. $2a^3 + 10a^2b - 3ab^2 - 15b^3$.
15. $2x^3 - x^2 - 8x + 4$.
16. $x^3 - 5x^2 - 36$.
17. $4x^3 + 16x^2 - x - 4$.

8. If $x - 1$ is a factor of $ax^3 + bx^2 + cx + d$, what condition st be satisfied by the coefficients? Make a general statement cerning the condition under which a polynomial is divisible $x - 1$.

9. Show that $3x + 4$ is a factor of $3x^3 + 4x^2 + 6x + 8$, and l the other factor.

0. If $x^3 + 2x^2 - x + k$ is divisible by $x - 2$, find k.

1. If $3x^3 - mx + 5$ is divisible by $x + 1$, find m.

2. State the converse of the theorem in the box on page 92. es the truth of the converse follow from that of the theorem? he converse true?

3. State the converse of the theorem in the box on page 93. es the truth of the theorem guarantee the truth of the con- se? Is the converse true?

Summary

The polynomials factored, so far, have been convenient- separated into exercises involving for the most part only method of factoring. The real test of your ability in oring is your skill in dealing with mixed sets of exercises in- ving all the types of the chapter.

Before attacking such a set, it is advisable to review the racteristics of the standard types and the methods of factoring m. The following table may be helpful in recalling the salient s.

Type	*Suggested by*	*Method*
Common factor	Factor common to every term	(C.F.) (second factor)
Trinomial	Three terms	Judicious guessing
Difference of squares	$(\quad)^2 - (\quad)^2$	Use the formula: $x^2-y^2 = (x + y)(x - y)$
Grouping	Four or more terms	Group for a common factor; group squares to get $(\quad)^2 - (\quad)^2$.
Incomplete squares	At least fourth power; two terms are perfect squares; product term defective	Complete square to get $(\quad)^2 - (\quad)^2$.
Sum and difference of cubes	$(\quad)^3 \pm (\quad)^3$	Use formulas: $x^3+y^3=(x+y)(x^2-xy+y^2)$ $x^3-y^3=(x-y)(x^2+xy+y^2)$
Factor Theorem	Any polynomial, usually those not yielding to other methods.	If polynomial vanishes w $x=a$, then $x-a$ is a fac

The following examples are added to illustrate the man in which various methods are sometimes used in combination

Example 1. Factor: $6mnx^3 - n^2x^3 - 6mny^3 + n^2y^3$.

We have

$$x^3(6mn-n^2) - y^3(6mn - n^2) = (6mn - n^2)(x^3 - y^3)$$
$$= n(6m - n)(x - y)(x^2 + xy +$$

The method involved grouping, taking out common fact and factoring the difference of cubes.

mple 2. Factor: $a^3 + b^3 - (c^2 + ab)(a + b)$.

We have

$$\begin{aligned}&(a + b)(a^2 - ab + b^2) - (c^2 + ab)(a + b)\\ &= (a + b)(a^2 - ab + b^2 - c^2 - ab)\\ &= (a + b)(a^2 - 2ab + b^2 - c^2)\\ &= (a + b)(a - b + c)(a - b - c).\end{aligned}$$

What methods of factoring were used in this solution?

mple 3. Factor:

$$3x^2 - 2xy - 5y^2 - 9ax^2 + 25ay^2 - 3x + 5y.$$

We have

$$\begin{aligned}&(3x^2 - 2xy - 5y^2) - a(9x^2 - 25y^2) - (3x - 5y)\\ &= (3x - 5y)(x + y) - a(3x + 5y)(3x - 5y) - (3x - 5y)\\ &= (3x - 5y)(x + y - 3ax - 5ay - 1).\end{aligned}$$

EXERCISES ON CHAPTER IV

A

tor fully:

1. $4x^2 - 20xy$.
2. $3a^2 - 14a + 15$.
3. $n^2 - 100$.
4. $(m + n)x + (m + n)y$.
5. $m^3 + 1000$.
6. $4x^2 - 9$.
7. $y^3 - 1$.
8. $p^2 - 4pr + 4r^2$.
9. $8a^3 - 27$.
10. $1 + 125r^3$.
11. $x^2 - 11x + 24$.
12. $m^2n^3 - m^3n^2$.
13. $3b^2 - 12$.
14. $k^2 - k - 42$.
15. $4m^2n^2 - 1$.
16. $(a - 2b)x - (a - 2b)y$.
17. $2r^3 - 16$.
18. $2a^2 - 11a - 21$.
19. $(m + n)^2 - k^2$.
20. $r^2 - 2rs + s^2 - t^2$.
21. $81x^2 - 49y^2$.

B

22. $ax - ay + bx - by$.
23. $x^4 - 3x^2y^2 + y^4$.
24. $a^3 - 4a^2 + a + 6$.
25. $x^2 - 4y^2 + x + 2y$.
26. $ax^4 - a$.
27. $2xy - x^2 - y^2 + 1$.
28. $(2m^2 - 5)^2 - (2m^2 - 5) - 6$.

29. $(x - y)^3 - 3(x - y)^2 - x + y$.

30. $(2k - n)^3 - (2k - n)^2$.

31. $a^2 - 4ab + 4b^2 - a + 2b$.

32. $24x^2y^6z - 375x^2y^3z^4$.

33. $9a^4 - 37a^2 + 4$.

34. $3x^3 + 9x^2 - 5x - 15$.

35. $x^3 - 4x + 15$.

36. $121x^4 + 216x^2y^2 + 100y^4$.

37. $4x^2 - 9y^2 + 2x - 3y$.

38. $2x^2 + xy - 6y^2 - x - 2y$.

39. $6a^3 - 9a^2 + 4a - 6$.

40. $8a^3 + 27b^3 - 4a^2 + 9b^2$.

41. $60x^2 - 71xy - 40y^2$.

42. $axym + 2axyn - bxwm - 2bxwn$.

43. $24a^4b + 94a^3b^2 - 8a^2b^3$.

44. $(m - 3n)^3 + 27n^3$.

45. $r^3 - 27s^3 + r^2 - 9s^2 - r + 3s$.

46. $(3x + 5y - 2z)^2 - (x - 5y + 2z)^2$.

47. $32 - 4m^3 - 8n^2 + m^3n^2$.

48. $(x^2 - 2x)^2 - 2(x^2 - 2x) - 3$.

49. $x(2m - n) + 2mn - x^2$.

50. $a^4b^2 + 6a^2b + 9$.

51. $a^3 + 3a^2 + 3a + 1$.

52. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.

53. $m^2n^2 - 4mnk + 4k^2$.

54. Use factoring to show that $(5k - 7m + 3n)^3 + (7m - 5k)^3 = 0$.

55. Use factoring to divide the product of $6x^2 + 13x -$ and $8x^2 + 15x - 2$ by $24x^3 + 13x^2 - 66x + 8$.

56. Find one factor of $(5a - 3b + 7c)^3 - (4a - 4b + 6c)$

57. Show that $x + 1$ is a factor of $3x^5 + 2x^4 - x^3 - 5x$ $7x + 12$.

58. If $x^3 + 7x^2 - 5x + k$ is divisible by $x - 1$, find k.

C

59. If $x^3 + 3x^2 + kx + m$ is divisible by $x + 1$ and x find k and m.

60. Factor fully $4a^2c^2 - (a^2 + c^2 - b^2)^2$.

61. Factor fully (*a*) $64x^6 - 729y^6$;
(*b*) $a^2 + b^2 + 1 + 2a + 2b + 2ab$.

62. Factor fully $x^{12} + 64y^{12}$.

3. If $x - 2$ is a factor of $ax^3 + bx^2 + cx + d$, find the con-
ion that must be satisfied by the coefficients.

4. If $2x^3 + kx^2 - 11x + m$ is divisible by $x - 3$ and $x + 2$,
d k and m.

5. Factor $a^3 + b^3 + c^3 - 3abc$.

6. Use the Factor Theorem to show that (*a*) $a - b$ is a
tor of $a^3 - b^3$, (*b*) $a + b$ is a factor of $a^3 + b^3$.

7. Factor fully $x^{64} - 1$.

8. Construct a geometrical figure to illustrate the formula
$(a + b)^2 = a^2 + 2ab + b^2$; (*b*) $(a - b)^2 = a^2 - 2ab + b^2$;
(*c*) $(a - b)(a + b) = a^2 - b^2$.

9. Find a condition on a, b, and c which will ensure that

$$a^2x^4 + bx^2y^2 + c^2y^4$$

be factored by the method of incomplete squares.

CHAPTER V

APPLICATIONS OF FACTORING: FRACTIO

Common Factors

Each of the numbers 24, 36, 48 has the factor 2; thus 2 i
common factor of the three numbers. In the same w
3, 4, 6, 12 are also common factors of 24, 36, 48. Since 12 is
largest common factor, it is called the ***highest common fac***
(H.C.F.) of the three given numbers.

You are already familiar with the extension of these ideas
algebra. We used the notion of common factors in Chapter
Thus it is seen that

x^2y is a common factor of $3x^4y^2$ and $7x^2yz$, and

$a - b$ is a common factor of $a^2 - b^2$ and $(a - b)^2$.

The idea of highest common factor in algebra is different fr
that used in arithmetic. Consider the terms

$$3x^4y^2 \text{ and } 7x^2yz.$$

The following are common factors of these terms:

$$x,\ y,\ xy,\ x^2,\ x^2y.$$

We say that x^2y is the H.C.F. since it is the common facto
highest degree.

NOTE: x^2y is not necessarily the *largest* common factor; if, for exan
$x = 1/2$ and $y = 1/3$, then x^2y has the *smallest* value of all the com
factors listed above. The important thing is that its degree is highes
arithmetic, however, the important idea is not conveyed by the word *hi*
but by the word *greatest*; thus, 12 is the greatest common factor of 24, 36
For this reason, there is merit in using the term *greatest common measur*
integers, and the term *highest common factor* for polynomials.

The highest common factor (H.C.F.) of two or more polynomials is their common factor of highest degree.

If the literal expressions have integral coefficients, their C.F. is the highest common factor of their literal parts times t of their integral parts. Common factors and highest com- n factors required in this chapter can be written down by ection once the expressions are in factored form.

mple 1. Write three common factors and the H.C.F. of

$$6x^2y^3,\ 12x^3y^2,\ 18x^3y^3.$$

By inspection we obtain $2y$, $3x^2$, and $6xy^2$ as common factors of the three given expressions; other common factors might be written. Since 6 is the largest common factor of the integral coefficients, and x^2y^2, the common factor of the literal parts with highest degree, we get the

$$\text{H.C.F.} = 6x^2y^2.$$

Test the result by dividing $6x^2y^2$ into each of

$$6x^2y^3,\ 12x^3y^2, \text{ and } 18x^3y^3.$$

mple 2. Find the H.C.F. of

$$6(x^2 - y^2),\ \ 8(x^3 - y^3),\ \ 4(x^2 - 2xy + y^2).$$

We have

$$6(x^2 - y^2) = 6(x + y)\,(x - y),$$
$$8(x^3 - y^3) = 8(x - y)\,(x^2 + xy + y^2),$$
$$4(x^2 - 2xy + y^2) = 4(x - y)^2.$$

Therefore, the H.C.F. $= 2(x - y)$.

EXERCISES

For oral answers

1. Which of the following are common factors of 18, 48, 54: , 4, 5, 6, 8, 9, 18? What is the G.C.M. of 18, 48, 54?

2. Which of the following are common factors of $4a^2bc$, c, $8a^3b^2$: $2a$, ab, a^2b, $2ab$, $4a^2$, $2abc$? What is the H.C.F. of given expressions?

e three common factors and the H.C.F. of each of the wing:

3. 36, 54, 144.
4. $6mn^2$, $24m^2n^3$, $72m^3n^4$.
5. a^3b^3c, $4a^2b^3c^3$, $10ab^3c^5$.
6. $8k^2mn$, $12kmn^2$.

7. $5x^2y^3$, $10x^3y$, $15x^2y^2z^2$.
8. $6p^2qr^2$, $9p^3q^2r^2$, $12p^3q^3r^3$.
9. $40a^5b^4c^3$, $48a^7b^5c^2$.
10. $22x^9y^5$, $55x^3y^2$.

For written answers

Find the H.C.F. of each of the following sets of expressions

11. $x^2 - y^2$, $x^2 - 2xy + y^2$, $x^3 - y^3$.
12. $a^2 - a - 20$, $a^2 - 25$, $2a^2 - 9a - 5$.
13. $4m^2 - 64$, $2m^2 + 16m + 32$, $10m + 40$.
14. $6x^4 - 6x^3y$, $9x^3(x^2 - 2xy + y^2)$.
15. $4x^2 - 9y^2$, $8x^3 + 27y^3$, $2x^2 + xy - 3y^2$.
16. $a^4 + a^2b^2 + b^4$, $a^3 - b^3$, $a^3 + a^2b + ab^2$.
17. $6x^2 - 4x - 10$, $12x^2 - 26x + 10$, $36x^2 - 100$.
18. $x^3 - 2x^2 + 7x - 14$, $2x^3 - x^2 - 5x - 2$.
19. $x^4 - 13x^2 + 36$, $x^2 + 5x + 6$, $x^3 + 4x^2 + x - 6$.
20. $25c^3 - cd^2$, $10c^3 + 13c^2d - 3cd^2$, $125c^4 - cd^3$.

Reducing Fractions to Lowest Terms

It is often a waste of labour to work with fractions like

$$\frac{17}{34}, \frac{120}{180}, \frac{a^2b}{ab^2}, \frac{x+y}{x^2-y^2}.$$

Such fractions can be simplified by using a basic principle wh is of first importance in dealing with fractions. You have r and used this principle many times. We recall it for re reference.

> *The terms of a fraction can be multiplied or divided by th same number* ($\neq 0$) *without changing the value of the fraction*

This means that ***factors*** common to both numerator denominator of a fraction can be divided out. If we divide b terms by their H.C.F., the fraction is reduced to its lowest te in one operation. However, in reducing fractions to their lo

is, it is often more convenient to factor the terms and then de out their common factors in succession, without finding r H.C.F.

mple 1. Reduce to its lowest terms:

$$\frac{(a^2 - 2ab + b^2)\,(a^2 + 2ab + b^2)}{a^4 - b^4}.$$

Factoring the terms, we have

$$\frac{(a-b)^2\,(a+b)^2}{(a^2-b^2)\,(a^2+b^2)} = \frac{(a-b)\;(a+b)}{\underset{1}{\cancel{(a-b)}}\;\underset{1}{\cancel{(a+b)}}\,(a^2+b^2)} = \frac{a^2-b^2}{a^2+b^2}.$$

mple 2. Reduce to its lowest terms: $\dfrac{x^4 + x^2y^2 + y^4}{x^6 - y^6}$.

The numerator can be factored as an incomplete square, nd the denominator as the difference of squares followed by he sum and difference of cubes. This gives

$$\frac{(x^2+y^2)^2 - x^2y^2}{(x^3-y^3)\,(x^3+y^3)} = \frac{\underset{1}{\cancel{(x^2+y^2+xy)}}\;\underset{1}{\cancel{(x^2+y^2-xy)}}}{(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)} = \frac{1}{x^2-y^2}.$$

EXERCISES

uce each of the following fractions to its lowest terms:

1. $\dfrac{a^2 - b^2}{a^3 - b^3}$.

2. $\dfrac{m^2 - m - 12}{m^2 + 2m - 24}$.

3. $\dfrac{4x^2 - 9y^2}{2x^2 + xy - 3y^2}$.

4. $\dfrac{4x^3y - 4x^2y^2 + 4xy^3}{x^3y^2 + y^5}$.

5. $\dfrac{z^2 - z - 2}{z^3 - 2z^2 + z - 2}$.

6. $\dfrac{x^2 + 2xy + y^2 - z^2}{x^2 - y^2 - 2yz - z^2}$.

7. $\dfrac{x^3 + 2x^2 - 13x + 10}{x^3 - 6x^2 + 11x - 6}$.

8. $\dfrac{4x^3 - 12x^2 - 9x + 27}{2x^3 - 9x^2 + 9x}$

9. $\dfrac{p^4 + 4q^4}{qp^2 + 2q^2p + 2q^3}$.

10. $\dfrac{5ab - 10ac - 3bd + 6cd}{6b^2 - 7bc - 10c^2}$

11. $\dfrac{72c^3d - 98cd^3}{6c^2d - d^2c - 7d^3}$.

12. $\dfrac{(m^4 - n^4)(m^2 - mn + n^2)}{m^6 - n^6}$

Multiplication and Division of Fractions

No new methods are involved in this section. Recall th

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

The product of two fractions is the product of their numerators divided by the product of their denominators.

This rule for multiplication means that when fractions are multiplied they "fuse" into a single fraction whose terms ha the corresponding terms of the original fractions as ***factc***

Thus, since

$$\frac{a}{b} \cdot \frac{b}{c} = \frac{ab}{bc},$$

we can divide out the factor b ***before*** multiplying:

$$\frac{a}{\cancel{b}} \cdot \frac{\overset{1}{\cancel{b}}}{c} = \frac{a}{c}.$$
$$\underset{1}{}$$

This is usually done in practice.

NOTE. ***Beware*** *of confusing the multiplication problem* $\frac{a}{b} \cdot \frac{b}{c}$ *with the addition problem* $\frac{a}{b} + \frac{b}{c}$. *In the addition process, the b may* ***not*** *be divided out; the two fractions are combined by addition in a very different manner from that used in multiplication.*

Example 1. Multiply:

$$\left(\frac{4x^2 - 12xy + 9y^2}{4x^2 - 9y^2}\right)\left(\frac{2x^2 + xy - 3y^2}{5x^2 - 9xy + 4y^2}\right).$$

Factoring and applying the basic principle, we have

$$\frac{\overset{2x-3y}{\cancel{(2x-3y)^2}}}{\underset{1}{\cancel{(2x+3y)}}\,\underset{1}{\cancel{(2x-3y)}}} \cdot \frac{\overset{1}{\cancel{(2x+3y)}}\,\overset{1}{\cancel{(x-y)}}}{(5x-4y)\,\underset{1}{\cancel{(x-y)}}} = \frac{2x-3y}{5x-4y}.$$

Division by a fraction is readily changed into a corresponding problem in multiplication. Consider the division of the number by $\frac{3}{4}$. Since $n \div \frac{3}{4}$ can be written $\frac{n}{\frac{3}{4}}$, we can multiply the terms by 4 and get

$$\frac{n}{\frac{3}{4}} = \frac{4n}{3} = n \times \tfrac{4}{3}.$$

To divide a number by a fraction, multiply the number by the reciprocal of the fraction.

Example 2. Simplify:

$$\left(\frac{m^4 + m^2n^2 + n^4}{m^3 - n^3}\right)\left(\frac{m^2 - n^2}{m^2 - 7mn + 10n^2}\right) \div \left(\frac{m^3 + n^3}{m^2 - 10mn + 25n^2}\right).$$

We factor the terms, replace the third fraction by its reciprocal, and apply the basic principle.

$$\frac{1}{\cancel{(m^2+n^2+mn)}\cancel{(m^2+n^2-mn)}}{\cancel{(m-n)}\cancel{(m^2+mn+n^2)}} \cdot \frac{1}{\cancel{(m-n)}\cancel{(m+n)}}{\cancel{(m-5n)}\,(m-2n)} \cdot \frac{m-5n}{\cancel{(m-5n)^2}}{\cancel{(m+n)}\cancel{(m^2-mn+n^2)}} = \frac{m-}{m-}$$

EXERCISES

Simplify the following expressions by performing the multiplications and divisions indicated.

1. $\dfrac{x^2-8x+15}{x^2-11x+30} \cdot \dfrac{x^2-3x-18}{x^2-5x+6}$.

2. $\dfrac{m^2-n^2}{(m+n)^2} \cdot \dfrac{m^3+n^3}{m^3-n^3}$.

3. $\dfrac{2x^2y-2xy^2}{3x^2-xy-2y^2} \cdot \dfrac{9x^2-4y^2}{9x^2-12xy+4y^2}$.

4. $\dfrac{8x^2y^3}{12x^3y^2} \cdot \dfrac{x-3xz+4xz^2}{7-21z+28z^2}$.

5. $\dfrac{a-3}{a+2} \cdot \dfrac{a^2-9a-22}{a^2-10a+21} \div \dfrac{a-11}{a^2-49}$.

6. $\dfrac{m^2-mn}{m+n} \cdot \dfrac{m^2+2mn+n^2}{m^2+mn} \div \dfrac{m^2-n^2}{m^2+n^2}$.

7. $\dfrac{27p^3-q^3}{27p^3+q^3} \div \dfrac{(3p-q)^2}{9p^2-q^2}$.

8. $\dfrac{x+7y}{x^2+9xy} \cdot \dfrac{x^3+9x^2y}{xy+7y^2}$.

9. $\dfrac{k^2-2km+m^2-n^2}{k^2-2kn+n^2-m^2} \div \dfrac{n^2-k^2-m^2+2km}{m^2-k^2-n^2+2kn}$.

10. $\dfrac{3a^3}{5bc} \cdot \dfrac{5b^3}{7ca} \div \dfrac{3ab}{7c^3}$.

11. $\dfrac{6c^2 + cd - 12d^2}{10c^2 + 13cd - 3d^2} \cdot \dfrac{5c^2 + 9cd - 2d^2}{9c^2 - 16d^2} \cdot \dfrac{27c^3 + 64d^3}{(c + 2d)^3}$.

12. $\dfrac{a^4 + a^2b^2 + b^4}{a^4 - a^2b^2 + b^4} \cdot \dfrac{a - b}{a + b} \div \dfrac{a^6 - b^6}{a^6 + b^6}$.

13. $\dfrac{6x^2 - bx - 2b^2}{bx - b^2} \cdot \dfrac{x^2 - bx}{9x^2 - 4b^2} \div \dfrac{2x + b}{3bx + 2b^2}$.

14. $\dfrac{3x^3 + 6x^2 + 4x + 8}{2x^3 + 4x^2 - 5x - 10} \div \dfrac{9x^4 - 16}{4x^4 - 25}$.

15. $\dfrac{m^2 + 2mn + n^2 + km + kn}{m^2 - (k + n)^2} \cdot \dfrac{m^2n - kmn - mn^2}{kmn + kn^2}$.

16. $\dfrac{25x^2 + 16y(4y - 5x)}{25x^2 - 64y^2} \cdot \dfrac{x + 2y}{5x + 8y} \div \dfrac{10x^2 - 11xy - 8y^2}{25x^2 + 16y(4y + 5x)}$.

Common Multiples

Let us recall the meaning of ***multiple.*** Since 4 is a ***factor*** 12, we say that 12 is a ***multiple*** of 4. Similarly, 12 is a multiple of 3. Hence 12 is a ***common multiple*** of 3 and 4. The numbers 3 and 4 have common multiples without end; for example, 24, 36, 48, 240, 3600, and so on. Since 12 is the smallest common multiple of 3 and 4, it is called the ***least common multiple*** (L.C.M.).

Example 1. (a) The L.C.M. of 25, 75, 100 can be found by selecting their factors.

$$25 = 5 \times 5,$$
$$75 = 5 \times 5 \times 3,$$
$$100 = 5 \times 5 \times 4.$$

The L.C.M. is 300 since it must have the factors 5, 5, 3, 4.

(b) The numbers 5, 12, 15 have, among others, t common multiples 120, 240, and 900. However, the L.C. of 5, 12, 15 is 60.

Suggestion: Check each L.C.M. mentally by dividing int each of the given numbers. For example, each of 25, 75, a 100 divides evenly into 300.

The foregoing ideas are readily extended to the numbers algebra. Since b is a factor of ab, then ab is a multiple of b. T numbers x^2y and xy^2 have for common multiples x^2y^2, x x^3y^4, $x^{20}y^{20}$, and so on. Since x^2y^2 is the common multiple ***lowest degree,*** it is called the ***lowest common multi*** (L.C.M.) of x^2y and xy^2.

> *The lowest common multiple (L.C.M.) of two or more polynomials is their common multiple of lowest degree.*

NOTE: The L.C.M. of a number of polynomials is not necessarily the *sm* common multiple; if, in the preceding example, $x = 1/2$ and $y = 1/3$, t of the multiples mentioned, x^2y^2 is the largest in magnitude as well as lowest in degree. As with the H.C.F., the degree is the important idea the word *lowest* is the pertinent adjective.

On the other hand, the L.C.M. of integers is the ***smallest*** common tiple, and the word ***least*** is correctly used to qualify the common mult

To find the L.C.M. of a number of polynomials with inte factors, we take the product of the least common multiple of integers and the lowest common multiple of the polynomial

Example 2. Find two common multiples and the L.C.M. $3ab$, $4a^2b$, $6abc$.

Since each of the given numbers is a factor of $24a^2$ this number is a common multiple of the given numb Similarly, $240a^3b^3c^3$ is a common multiple, and we could others at pleasure.

The least common multiple of 3, 4, and 6 is 12, and lowest common multiple of ab, a^2b, and abc is a^2bc. Theref the L.C.M. is $12a^2bc$.

For more complicated expressions, common multiples and the L.M. may be discovered by inspecting the factors of the expressions, keeping in mind the meanings involved.

mple 3. Find the L.C.M. of

$$6(x^2 - y^2),\ 8(x^3 - y^3),\ 4(x^2 - 2xy + y^2).$$

Factoring the expressions, we obtain

$$6(x^2 - y^2) = 6(x + y)(x - y),$$
$$8(x^3 - y^3) = 8(x - y)(x^2 + xy + y^2),$$
$$4(x^2 - 2xy + y^2) = 4(x - y)^2.$$

The L.C.M. of 6, 8, and 4 is 24. The L.C.M. of the literal parts must include the factors $x + y$, $x - y$, $x^2 + xy + y^2$, and $(x - y)^2$. Therefore,

$$\text{L.C.M.} = 24(x + y)(x - y)(x^2 + xy + y^2)(x - y)$$
$$= 24(x^2 - y^2)(x^3 - y^3).$$

EXERCISES

For oral answers

. Which of the following are common multiples of 6, 15, 60, 120, 180, 240, 300, 2160? What is the L.C.M. of 6, 24?

. Which of the following are common multiples of $2x$, $3xy$, $24x^3y$, $30x^2y^2$, $36xy^3$, $48x^3y^3$, $120x^5y$? What is the L.C.M. of $2x$, $3xy$, and $4xy^2$?

e two common multiples and the L.C.M. of each of the wing sets of numbers:

. 5, 20, 50.

. m, n, m^2n^2.

. $4p^2$, $9q^2$, $12p^3$.

. $30k^2m^3$, $24m^2n^2$.

4. 8, 18, 72.

6. $3ab$, $6a^3b$, $12ab^3$.

8. $6xy$, $10yz$, $5zx$.

10. $abcd$, $8a^2bc$, $20b^2cd$.

For written answers

-20. Find the L.C.M. of each of the sets of expressions in cises 11-20 on page 102.

Find the L.C.M. of each of the following sets of expressions

21. $x^4 - y^4$, $x^2 - y^2$, $(x - y)^2$, $(x + y)^2$.
22. $x^3 + 5x^2 - 17x - 21$, $2x^3 - 5x^2 - 4x + 3$.
23. $x^2 - (y + z)^2$, $(x + y)^2 - z^2$, $x^2 + y^2 + z^2 + 2xy + 2yz +$
24. $49a^2b^2 - 100c^2$, $49a^2b^2 - 140abc + 100c^2$.
25. $4x^4 - 37x^2 + 9$, $2x^2 - 5x - 3$, $6x^2 + 15x - 9$.
26. $m - 1$, $m^2 - 1$, $m^3 - 1$, $m^4 + m^2 + 1$.
27. $9p^2 - 4q^2 + 20q - 25$, $3p^2 + 2pq - 5p$.
28. $3a^3 + 6a^2 + 4a + 8$, $2a^3 + 4a^2 - 5a - 10$.
29. $6c^4d - 2c^3d^2 - 20c^2d^3$, $24c^3d^2 + 28c^2d^3 - 20cd^4$.

Addition and Subtraction of Fractions

The purpose of the present section is not to introduce principles but rather to review and consolidate basic meth that have been encountered in earlier courses.

Recall that fractions with the same denominator can be ad and subtracted without difficulty. If fractions with diffe denominators are to be added or subtracted, the basic princ on page 102 comes once more to our aid. Using this princi we can make the denominators alike without changing values of the fractions.

Example 1. Add: $\frac{1}{3} + \frac{1}{2}$.

We have $\frac{1}{3} + \frac{1}{2} = \boxed{\frac{2}{6} + \frac{3}{6}} = \frac{2 + 3}{6} = \frac{5}{6}$.

This simple problem is worth study. The part of the solu in the box, although usually omitted, is the cornerstone of operation. It is the point at which we use the basic princip obtain the common denominator 6; the terms of 1/3 are tiplied by 2, and the terms of 1/2 are multiplied by 3. The s principle is used in adding or subtracting complicated fracti

suggested that you omit the step in the box only when you ure that its meaning is understood.

nple 2. Simplify by addition and subtraction:

$$\frac{2x}{x^2 - y^2} + \frac{1}{x - y} - \frac{1}{x + y}.$$

In choosing a common denominator, we may use any ommon multiple of $x^2 - y^2$, $x - y$, and $x + y$. However, ne computation is simplified if we use the L.C.M. which $x^2 - y^2$.

Therefore, to make all the denominators equal $x^2 - y^2$, e multiply the terms of the second fraction by $x + y$, and ne terms of the third by $x - y$. We get

$$\frac{2x}{(x + y)(x - y)} + \frac{x + y}{(x + y)(x - y)} - \frac{x - y}{(x + y)(x - y)}$$

$$= \frac{2x + x + y - (x - y)}{x^2 - y^2} = \frac{2\cancel{(x + y)}}{\cancel{x^2 - y^2}} = \frac{2}{x - y}.$$

The L.C.M. of the denominators of a number of fractions is called the ***least common denominator (L.C.D.)*** *of the fractions.*

ıple 3. Subtract:

$$\frac{2a^2 + ab}{2a^2 + 3ab + b^2} - \frac{6ab - 2b^2}{3a^2 + 5ab - 2b^2}.$$

Factoring, we have

$$\frac{1}{\cancel{a(2a+b)}\ (a+b)} - \frac{1}{2b\cancel{(3a-b)}\ (a+2b)} = \frac{a}{a+b} - \frac{2b}{a+2}$$

$$= \frac{a(a+2b)}{(a+b)(a+2b)} - \frac{2b(a+b)}{(a+2b)(a+b)}$$

$$= \frac{a^2+2ab-2ab-2b^2}{(a+b)(a+2b)}$$

$$= \frac{a^2-2b^2}{(a+b)(a+2b)}$$

This example suggests that *before finding the L.C.D., it is often advisable to reduce the fractions to their lowest terms.*

EXERCISES

Perform the indicated additions and subtractions in each of the following:

1. $\frac{1}{x}+\frac{1}{y}$.
2. $\frac{3}{b}-\frac{5}{bc}$.
3. $\frac{5}{6}+\frac{7}{15}$.
4. $\frac{5}{xy}+\frac{7}{yz}$.
5. $\frac{13}{18}-\frac{7}{12}+\frac{11}{30}$.
6. $\frac{13}{xy}-\frac{7}{xz}+\frac{11}{xa}$.
7. $\frac{a-b}{2}+\frac{a-b}{6}-\frac{2a-b}{8}$.
8. $\frac{3}{m+n}-\frac{2}{m-n}$.
9. $\frac{y^2}{x^2-y^2}+\frac{y}{x+y}$.
10. $\frac{a}{a+b}-\frac{a^2}{(a+b)^2}$.

1. $\dfrac{x}{x^3 - y^3} - \dfrac{1}{x^2+xy+y^2}$.

12. $\dfrac{1}{(p+q)^2} + \dfrac{1}{p^2 - q^2}$.

13. $\dfrac{r-s}{r+s} - \dfrac{r+s}{r-s}$.

14. $\dfrac{x-1}{x-2} + \dfrac{x-3}{x-4}$.

15. $\dfrac{1}{a-b} + \dfrac{1}{a+b} - \dfrac{2a}{a^2-b^2}$.

16. $\dfrac{x}{x^2 - x} + \dfrac{a}{ax + a}$.

17. $\dfrac{1}{a^2 + 3a + 2} + \dfrac{1}{a^2 + 4a + 3}$.

18. $\dfrac{a^2 - a - 2}{a^2 - 2a - 3} - \dfrac{a^2 - 5a + 6}{a^2 - 7a + 10}$.

19. $\dfrac{a+b}{a^2 + 2ab + b^2} + \dfrac{a-b}{a^2 - 2ab + b^2} - \dfrac{2a^2}{a^3 - ab^2}$.

20. $\dfrac{(2x+3)^2}{2x^2 + 13x + 15} + \dfrac{x^2 + x - 30}{x^2 - 10x + 25} - \dfrac{3x^2 + 4x}{x^2 - 25}$.

21. $\dfrac{2x}{8x^3 - 2x} + \dfrac{4x^2 + 8x + 3}{4x^2 + 4x - 3} - \dfrac{4}{4x - 2}$.

22. $\dfrac{m}{m-n} - \dfrac{2n-m}{m+n} - \dfrac{m^2+n^2}{m^2-n^2}$.

23. $\dfrac{2x^2 + 3x - 3}{8x^3 + 1} - \dfrac{3x - 4}{4x^2 - 2x + 1} + \dfrac{1}{2x + 1}$.

24. (*a*) $\dfrac{1}{a-b} + \dfrac{1}{a+b}$; (*b*) $\dfrac{1}{a-b} + \dfrac{1}{a+b} + \dfrac{2a}{a^2+b^2}$.

25. $\dfrac{1}{a-b} + \dfrac{1}{a+b} + \dfrac{2a}{a^2+b^2} + \dfrac{4a^3}{a^4+b^4}$.

26. $\dfrac{3x-1}{3x^2 + 7x - 6} - \dfrac{2(3x-2)}{2x^2 + 3x - 9} + \dfrac{x+3}{6x^2 - 13x + 6}$.

27. $\dfrac{x^2-y^2-z^2+2yz}{x^2+2xy+y^2-z^2}+\dfrac{y^2-z^2-x^2+2xz}{y^2+2yz+z^2-x^2}+\dfrac{z^2-x^2-y^2+2xy}{z^2+2zx+x^2-y^2}$.

28. $\dfrac{m}{5m-n}-\dfrac{m}{5m+n}-\dfrac{2mn}{25m^2+n^2}$.

29. $\dfrac{7x+2y}{7x-2y}-\dfrac{7x-2y}{7x+2y}-\dfrac{16y^2}{49x^2-4y^2}$.

30. $\dfrac{1}{a-1}-\dfrac{2}{a+2}+\dfrac{3}{a-3}-\dfrac{2}{a-4}$.

31. Write two fractions which have as their sum (*a*) $\dfrac{x+}{xy}$

(*b*) $\dfrac{2x+3y}{xy}$, (*c*) $\dfrac{3x+4y}{12xy}$, (*d*) $\dfrac{ax^2+by^2}{x^2y}$, (*e*) $\dfrac{2x+3}{x^2y}$

Further Practice with Fractions

In transforming expressions involving fractions, the follow devices are sometimes helpful.

Use of the multiplier -1. The basic principle of fracti permits us to multiply the terms of a fraction by -1. Thu

$$\frac{3}{4}=\frac{-3}{-4}, \quad \text{and} \quad \frac{-3}{4}=\frac{3}{-4}.$$

From the rule of signs for division (page 5), we have

$$\frac{-3}{4}=-\frac{3}{4}=\frac{3}{-4}.$$

It follows that *the value of a fraction remains unchanged*

(1) *if both of its terms are multiplied by -1, or*

(2) *if one term is multiplied by -1, and the sign before fraction is changed.*

These uses of -1 as a multiplier sometimes enable us to ch a fraction into a more convenient form.

mple 1.

(*a*) $$-\frac{m}{-2} = -\frac{-m}{2} = +\frac{m}{2}.$$

(*b*) $$-\frac{a-b}{y-x} = -\frac{b-a}{x-y} = +\frac{b-a}{y-x} = +\frac{a-b}{x-y}.$$

Note that $\boxed{-(b-a) = a-b}$

mple 2. Simplify: $\dfrac{1}{m-2n} - \dfrac{2}{m+2n} - \dfrac{3m}{4n^2-m^2}$.

For the purpose of finding the L.C.D., the denominator f the third fraction would be more conveniently written as $^2 - 4n^2$, which is $-(4n^2 - m^2)$. Therefore we multiply the enominator of the third fraction by -1 and change the sign efore the fraction. This gives

$$\frac{1}{m-2n} - \frac{2}{m+2n} + \frac{3m}{m^2-4n^2} = \frac{(m+2n)-2(m-2n)+3m}{m^2-4n^2}$$

$$= \frac{m+2n-2m+4n+3m}{m^2-4n^2} = \frac{2m+6n}{m^2-4n^2}.$$

mplex fractions. A fraction is called ***complex*** if it has ions in its numerator or its denominator or both. Complex ions may be reduced to simple fractions by using the basic iple: ***multiply the terms of the complex fraction by L.C.D. of the fractions which appear in its terms.***

mple 3. $\dfrac{\frac{5}{6}}{\frac{7}{9}} = \frac{15}{14}$. (Multiply the terms of the complex fraction by 18, the L.C.M. of 6 and 9.)

mple 4. Simplify: $\dfrac{\dfrac{1}{m+n} + \dfrac{2}{m-n}}{\dfrac{3}{m+n} - \dfrac{4}{m-n}}$.

The L.C.D. of the fractions in the terms of the com fraction is $m^2 - n^2$. Multiplying the terms of the com fraction by $m^2 - n^2$, we get

$$\frac{(m-n)+2(m+n)}{3(m-n)-4(m+n)} = \frac{3m+n}{-m-7n} = -\frac{3m+n}{m+7n}$$

Mixed expressions. An algebraic expression like

$$a + \frac{a}{a+b}$$

is analogous to a mixed number in arithmetic. Such expres are often manipulated more easily if they are first changed single fractions.

Example 5. Simplify: $\left(a + \frac{b^2}{a+b}\right) \div \left(1 - \frac{b^2}{a^3}\right)$.

Change each of the mixed expressions into a single tion by bringing it to a common denominator. We ha

$$\left\{\frac{a(a+b)+b^2}{a+b}\right\} \div \left\{\frac{a^3-b^3}{a^3}\right\} = \frac{a^2+ab+b^2}{a+b} \cdot \frac{a^3}{(a-b)(a^2+ab+b^2)}$$

$$= \frac{a^3}{a^2-b^2}.$$

Two popular pitfalls. Study the following and learn to the errors mentioned.

(1) *The reciprocal of the sum or the difference of two nu does* **not** *equal the sum or the difference of their reciprocals.* expressions like

$$1 - \frac{b^3}{a^3} \quad \text{and} \quad \frac{x}{y} + \frac{y}{x} - 1$$

st be transformed into single fractions before their reciprocals taken. What is the reciprocal of

$$\frac{1}{x}+\frac{1}{y}?$$

2) *The sum or the difference of two fractions is* ***not*** *obtained dding or subtracting their numerators and denominators.* Thus

$\frac{1}{2}+\frac{1}{3}$ is ***not*** equal to $\frac{1}{2+3}$, ***nor*** is $\frac{1}{5}-\frac{1}{4}$ equal to $\frac{1}{5-4}$;

$\frac{1}{a}+\frac{1}{b}$ is ***not*** equal to $\frac{1}{a+b}$, ***nor*** is $\frac{2}{a}-\frac{1}{b}$ equal to $\frac{1}{a-b}$.

at are the correct sums and differences?

EXERCISES

For oral answers

. Express each of the following as a fraction with no minus s in its terms: $(a)-\frac{-4}{5}$, $(b)-\frac{4}{-5}$, $(c)+\frac{-4}{-5}$, $(d)-\frac{-a}{n}$, $\frac{-c}{-d}$, $(f)-\frac{c}{-d}$.

. Express each of the following as a fraction with a plus in front:

$(a)\ -\frac{-a}{b}$, $(b)\ -\frac{-m}{-n}$, $(c)\ -\frac{x}{-y}$, $(d)\ -\frac{b-a}{c}$,

$(e)\ -\frac{x}{y-z}$, $(f)\ -\frac{m-n}{-k}$, $(g)\ -\frac{-c}{a-b}$,

$(h)\ -\frac{b-a}{y-x}$, $(i)\ -\frac{2}{p-q}$, $(j)\ -\frac{2(a-b)}{x-y}$.

3. Express each of the following as a fraction with a mi
sign in front:

(a) $\frac{-7}{6}$, (b) $\frac{d}{-c}$, (c) $\frac{a-b}{c}$, (d) $\frac{m}{k-n}$,

(e) $\frac{x}{y}$, (f) $\frac{-x}{y-z}$, (g) $\frac{x-y}{a-b}$, (h) $\frac{a(b-c)}{m-n}$,

(i) $\frac{5(y-z)}{-3(a-b)}$.

For written answers

Simplify:

4. $\dfrac{\frac{2}{3}}{\frac{5}{9}}$.

5. $\dfrac{\frac{x}{y}}{\frac{1}{y}}$.

6. $\dfrac{\frac{m}{n}}{\frac{5}{n^2}}$.

7. $\dfrac{\frac{2ab}{c^2}}{c}$.

8. $\dfrac{4x^2y}{\frac{x^3}{2y}}$.

9. $\dfrac{m+n}{\frac{1}{m}+\frac{1}{n}}$.

10. $\dfrac{\frac{2}{m+n}-\frac{1}{m-n}}{\frac{4}{m+n}+\frac{3}{m-n}}$.

11. $\dfrac{x+3+\frac{x^2-7x}{x+4}}{x-3-\frac{x^2-7x}{x+4}}$.

2. $\dfrac{3x-5}{5x-1}+\dfrac{2x+9}{1+5x}+\dfrac{25x^2}{1-25x^2}$.

3. $\dfrac{18}{4m-3n}+\dfrac{15}{3n-4m}$.

4. $\left(1+\dfrac{x}{y}\right)\left(1+\dfrac{y}{x}\right)\div\left(\dfrac{x}{y}-\dfrac{y}{x}\right)$.

5. $\left(\dfrac{x^2}{y}-\dfrac{y^2}{x}\right)\div\left(\dfrac{x}{y}+\dfrac{y}{x}+1\right)$.

6. $\dfrac{\dfrac{p+q}{p-q}-\dfrac{p-q}{p+q}}{\dfrac{p+q}{p-q}+\dfrac{p-q}{p+q}}$.

17. $\dfrac{a+\dfrac{b-a}{1-bc}}{1+\dfrac{c(b-a)}{1-bc}}$.

8. $\left(\dfrac{x^2+y^2}{2xy}+1\right)\left(\dfrac{x^2+xy+y^2}{2xy(x+y)}\right)\div\left(\dfrac{x^2}{y^2}-\dfrac{y}{x}\right)$.

9. $\dfrac{5}{3a-2b}+\dfrac{5}{2b+3a}+\dfrac{30}{4b^2-9a^2}$.

10. $\left(1-\dfrac{8x}{x^2+4x+3}\right)\left(2+\dfrac{8}{x-1}\right)\div\left(3+\dfrac{6}{x-5}\right)$.

11. $\dfrac{2}{p-q}-\dfrac{2(p-q)}{p^2+pq+q^2}+\dfrac{6q^2}{q^3-p^3}$.

12. $\dfrac{x-y+\dfrac{y^3}{x^2+xy+y^2}}{x-y-\dfrac{x^3}{x^2+xy+y^2}}$.

23. $$\left\{\frac{m+\dfrac{mn}{m-n}}{m^2+\dfrac{m^2n^2}{m^2-n^2}}\right\}\div\left\{\frac{\dfrac{1}{m^2}-\dfrac{1}{n^2}}{\dfrac{1}{m}-\dfrac{1}{n}}\right\}.$$

REVIEW EXERCISES ON CHAPTER V

A

1. What is meant by the H.C.F. of (*a*) two integers, (*b*) t polynomials?

2. Find the H.C.F. of (*a*) 54, 90, 126; (*b*) $6(m^2 -$ $4(m-n)^2$.

3. Reduce to its lowest terms:

$$\frac{3a^2x^2+2a^3x-a^4}{x^3-a^2x}.$$

4. Multiply: $$\left(\frac{x^2-6x+9}{x^2-9}\right)\left(\frac{x+3}{x^2+9}\right).$$

5. Simplify: $$\left(\frac{m^2+10m+25}{m^2-10m+25}\right)\div\left(\frac{2m^2+10m}{m^2-25}\right).$$

6. What is meant by the L.C.M. of (*a*) two integers, (*b*) polynomials?

7. Find the L.C.M. of (*a*) 45, 75, 105; (*b*) x^3+8, x^2+4x $3(x+2)$.

Simplify the following by performing the indicated additions subtractions:

8. $$\frac{2a}{a^2-9}+\frac{1}{a-3}-\frac{1}{a+3}.$$

9. $$\frac{11}{5k-4}-\frac{12}{4-5k}.$$

10. $$\frac{2}{a+1}+\frac{a-3}{a^2+a}-\frac{3a^2}{a^3-a}.$$

1. $\dfrac{x^2}{x^3 - y^3} - \dfrac{x + y}{x^2 + xy + y^2}$.

2. $\dfrac{2}{m^2 - 3m - 4} + \dfrac{1}{m^2 - 7m + 12}$.

B

plify the following by performing the indicated operations:

3. $\left(\dfrac{m^2}{n^2} - \dfrac{m}{n} + 1\right) \div \left(\dfrac{m^2}{n^2} + \dfrac{m}{n} + 1\right)$.

4. $\left(\dfrac{(a + b)^3}{a^3 + b^3}\right)\left(\dfrac{a^2 - ab + b^2}{a^2 + 2ab + b^2}\right)$. 1

5. $\dfrac{4}{z + 1} - \dfrac{z - 2}{z^2 + z} + \dfrac{3z}{1 - z^2}$.

6. $\left(\dfrac{x^6 + y^6}{x^6 - y^6}\right)\left(\dfrac{x - y}{x^2 + y^2}\right) \div \left(\dfrac{x^4 - x^2y^2 + y^4}{x^4 + x^2y^2 + y^4}\right)$. $= \dfrac{1}{x + y}$

7. $\dfrac{12c}{4c^2 - 1} + \dfrac{2}{1 - 2c} - \dfrac{3}{1 + 2c}$.

8. $\dfrac{2}{n^2 - 1} - \dfrac{2}{n^2 + 1} - \dfrac{4}{n^4 + 1}$. $\dfrac{8}{n^8 - 1}$

9. $\dfrac{a + b}{(b - c)(c - a)} + \dfrac{b + c}{(c - a)(a - b)} - \dfrac{c + a}{(b - a)(b - c)}$. $= 0$

10. $\dfrac{2}{x^2 - 1} - \dfrac{5}{3(x^2 + x + 1)} - \dfrac{2x^2 - 3x - 8}{3(1 - x^3)(x + 1)}$.

11. $\left(\dfrac{k}{1 + k} + \dfrac{1 - k}{k}\right) \div \left(\dfrac{k}{1 + k} - \dfrac{1 - k}{k}\right)$.

22. $\dfrac{6}{y^2+2y-8}+\dfrac{7}{12-y-y^2}-\dfrac{1}{5y-y^2-6}$.

23. $\dfrac{m^3-n^3}{(m+n)^3}\cdot\dfrac{m^2+2mn+n^2}{m^4+m^2n^2+n^4}\div\dfrac{m^2-2mn+n^2}{m^2-n^2}$.

24. $\dfrac{2}{\dfrac{1}{x}-\dfrac{1}{y}}+\dfrac{y}{1-\dfrac{y}{x}}-\dfrac{x}{1-\dfrac{x}{y}}$.

25. $\dfrac{r^2+s^2}{r^3-s^3}\cdot\dfrac{r^4+s^4}{r^4-s^4}\div\dfrac{r^8-s^8}{r^6-s^6}$.

26. $\left\{\dfrac{(a+b)^2}{4ab}-1\right\}\div\left\{\dfrac{a^2+b^2}{2ab}-1\right\}$.

27. $\dfrac{1}{x-1}-\dfrac{x+3}{x^2-3x+2}+\dfrac{5x}{x^3-6x^2+11x-6}$.

28. $\left(n+\dfrac{nx}{m}\right)\left(1-\dfrac{m}{m+x}\right)\left(\dfrac{m}{nx}\right)$.

29. $\left(\dfrac{1}{c+1}+\dfrac{1}{c-1}-\dfrac{2}{c+2}\right)\div\left(\dfrac{2}{c+1}-\dfrac{3}{c-1}+\dfrac{1}{c+2}\right)$.

30. $\dfrac{3x^2+5x-2}{14x^2+11x-15}\cdot\dfrac{7x^2-16x-15}{6x^2-11x+3}\div\dfrac{x^2-x-6}{4x^2-9}$.

31. $\dfrac{a^2-a+\dfrac{a-1}{a+1}}{a+\dfrac{1}{a+1}}$.

32. $\dfrac{x-1+\dfrac{x^2+6x}{x-5}}{x+1-\dfrac{x^2-4x}{x-5}}$.

33. $\dfrac{x+\dfrac{2-x}{1+2x}}{1-\dfrac{2x-x^2}{1+2x}}-\dfrac{2-\dfrac{2-x}{1-2x}}{1-\dfrac{4-2x}{1-2x}}$.

4. If $x = \dfrac{a-2}{a}$ and $y = \dfrac{a-2}{2}$, find the value of

$$x(y+2) + \frac{x}{y} + \frac{y}{x}.$$

5. If $x = \dfrac{a^2+b^2}{a+b}$, find the value of $\dfrac{a^2}{x-b} + \dfrac{b^2}{x-a}$.

C

6. Give a general proof of the fact that the sum of any two nbers equals their product times the sum of their reciprocals.

7. If $x = \dfrac{1-y}{1+y}$, find the value of $\dfrac{1-x}{1+x}$ in terms of y.

8. If $x = \dfrac{a}{1-a^2}$ and $a = \dfrac{b}{1-b^2}$, find x in terms of b.

9. Simplify: $\dfrac{1}{x-5} - \dfrac{1}{x+5} - \dfrac{10}{x^2+25} - \dfrac{500}{x^4+625}$.

10. Simplify: $\dfrac{(2a+3)^2}{2a^2+13a+15} + \dfrac{a^2+a-30}{a^2-10a+25} - \dfrac{3a^2+4a}{a^2-25}$.

11. If $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{f}$, find (*a*) f in terms of p and q, (*b*) p in terms f and f.

12. Substitute $a = 1 - \dfrac{1}{m}$ in the expression $\dfrac{a^2-1}{a^2+1}$ and plify.

43. If $x = m + n$, find the value of

$$\frac{1}{x^2 - m^2 + n^2} - \frac{1}{x^2 + m^2 - n^2} + \frac{1}{m^2 + n^2 - x^2}.$$

44. If $m = \dfrac{1}{x - y}$ and $n = \dfrac{1}{x + y}$, find the value of $\dfrac{m +}{m -}$

45. Simplify:

$$\frac{4x^2 - 9y^2}{2y^2 - 5xy - 3x^2} \cdot \frac{3x^2 + 11xy - 4y^2}{7xy - 6x^2 - 2y^2} \div \frac{2x^2 + 11xy + 12y}{4y^2 - x^2}$$

CHAPTER VI

INDICES

Some Uses of Powers

From the beginning of algebra, we have made much use of
h expressions as the following:

$$x^2,\ y^3,\ 5^2,\ 10^4,\ z^5,\ x^6,\ y^4,\ a^3,\ 3^3,\ m^8,\ y^{12}.$$

ese expressions are called ***powers***: for example, x^2 and x^6 are
wers of x; y^3, y^4, and y^{12} are ***powers of y***; 5^2 is a ***power of 5***;
d so on. In referring to a power, the words used can be il-
trated by

$$\boxed{x^3},$$

which x is called the ***base of the power*** and 3 (written as a
perscript) is called the ***index of the power.*** Sometimes we
er to the x and the 3 for short as the ***base*** and the ***index,***
pectively; the word ***exponent*** is often used as an alternative
index.

When you wish to point to an object, you use your *index* finger. There are many related uses of the word *index*. A cost-of-living index is a number which points out or indicates how the cost of living in a given year compares with that in some year taken as a standard. An index in a book points out the page or pages on which a given topic is mentioned. In algebra, the index of a power is a number which indicates the power to which the base is raised.

When the index of a power is a positive integer, it tells the number of times the base is to be used as a factor.

Powers of the same base can be combined compactly in
tain operations. Since these operations are not new, one

purpose of the present chapter is to recall them and to bri them into sharper focus. Once this has been done, we shall pr ceed to extend our notions about indices.

ORAL EXERCISES

(In the following exercises, m, n, and a are positive integers.)

1. What is the meaning of (*a*) x^3, (*b*) y^7, (*c*) 4^2, (*d*) x (*e*) x^{100}, (*f*) x^m, (*g*) y^a?

2. What is the value of (*a*) 2^3, (*b*) 1^8, (*c*) 0^2, (*d*) $(-1$ (*e*) $(-1)^4$, (*f*) $(-1)^{57}$, (*g*) $(-1)^{100}$, (*h*) $(-2)^3$, (*i*) $(-3)^5$?

3. If $z = 10$, find the value of (*a*) z^3, (*b*) $5z^2$, (*c*) $1000 \div$ (*d*) $4z^3 \div z^2$.

4. How many factors equal to x has the product $x^3 \cdot x$ Express this product as a power of x.

5. Express each of the following products as a power of x:
(*a*) $x^2 \cdot x^4$, (*b*) $x^6 \cdot x^{12}$, (*c*) $x^m \cdot x^4$, (*d*) $x^m \cdot x^n$.

6. How many factors equal to x has the quotient $x^5 \div$ Express this quotient as a power of x.

7. Express each of the following quotients as a power of
(*a*) $x^4 \div x^2$, (*b*) $x^{10} \div x^3$, (*c*) $x^m \div x^4$, (*d*) $x^m \div x^n$.

8. What is the meaning of $(y^3)^2$? How many factors involved? Express $(y^3)^2$ as a power of y.

9. Express each of the following as a power of x:
(*a*) $(x^4)^3$, (*b*) $(x^5)^2$, (*c*) $(x^m)^4$, (*d*) $(x^m)^n$.

10. What is the meaning of $(ab)^3$, of $(xy)^2$, and of $(mn$ Express without parentheses (*a*) $(xy)^{20}$, (*b*) $(xy)^a$, (*c*) $(xy)^m$.

11. What is the meaning of $\left(\frac{a}{b}\right)^3$ and of $\left(\frac{x}{y}\right)^2$? Express wi out parentheses (*a*) $\left(\frac{x}{y}\right)^{15}$, (*b*) $\left(\frac{x}{y}\right)^m$.

The Laws of Indices

The foregoing oral exercises lead to a number of generaliz-ons. Thus, if m is a positive integer, we get the following x^m:

$$x^m = x \cdot x \cdot x \cdot \cdots \text{ to } m \text{ factors.}$$

ain if m and n are positive integers and m is greater than certain operations with powers can be generalized as follows:

$$\begin{aligned}
&(1) \ldots x^m \cdot x^n = x^{m+n} \\
&(2) \ldots x^m \div x^n = x^{m-n} \\
&(3) \ldots (x^m)^n = x^{mn} \\
&(4) \ldots (xy)^m = x^m y^m \\
&(5) \ldots \left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}
\end{aligned}$$

These are called the ***Index Laws.*** While the process of neralization by which we arrived at them seems quite justified, s well to note that these laws can be readily proved by using definition of x^m. We shall illustrate the method of proof by ablishing the first three laws; the proofs of the last two laws left as exercises.

w 1: ***The Law of Multiplication***

By definition, we have

$$\begin{aligned}
x^m \cdot x^n &= (x \cdot x \cdot x \cdots \text{to } m \text{ factors}) \ (x \cdot x \cdot x \cdots \text{ to } n \text{ factors}) \\
&= x \cdot x \cdot x \cdots \text{ to } (m+n) \text{ factors} \\
&= x^{m+n}
\end{aligned}$$

is means that ***to obtain the product of two powers of the me base,*** we should

(1) *write the common base,* and
(2) *write as index the* ***sum*** *of the given indices.*

COROLLARY: $x^m \cdot x^n \cdot x^p = x^{m+n} \cdot x^p = x^{m+n+p}$.

Example. $x^2 \cdot x^3 \cdot x^5 = x^{2+3+5} = x^{10}$.

LAW II: ***The Law of Division***

By definition, we have

$$\begin{aligned} x^m \div x^n &= \frac{x \cdot x \cdot x \cdots \text{ to } m \text{ factors}}{x \cdot x \cdot x \cdots \text{ to } n \text{ factors}} \\ &= x \cdot x \cdot x \cdots \text{ to } (m-n) \text{ factors} \qquad (m > n) \\ &= x^{m-n}. \end{aligned}$$

Therefore, ***to obtain the quotient of two powers of th same base,*** we should

(1) *write the common base,* and
(2) *write as index the* ***difference*** *obtained by subtracting t index of the divisor from that of the dividend.*

NOTE: If $m < n$, we can proceed as follows:

$$\frac{x^3}{x^7} = \frac{1}{x^{7-3}} = \frac{1}{x^4}.$$

That is, we divide three x's from both numerator and denom inator. In general, where m is less than n, we can divide factors x from both numerator and denominator. This give

$$x^m \div x^n = 1/x^{n-m}.$$

LAW III: ***The Law of Powers***

As before, we have

$$\begin{aligned} (x^m)^n &= x^m \cdot x^m \cdot x^m \cdots \text{to } n \text{ factors} \\ &= (x \cdot x \cdot x \cdots \text{to } m \text{ factors})\,(x \cdot x \cdot x \cdots \text{ to } m \text{ factors}) \cdot \\ &\quad \cdots \text{repeated to } n \text{ sets of } m \text{ factors} \\ &= x \cdot x \cdot x \cdots \text{ to } mn \text{ factors} \\ &= x^{mn}. \end{aligned}$$

erefore, ***to raise a power to a power,*** we should

(1) *write the base*, and

(2) *write as index the* ***product*** *of the two indices.*

mple. From Law III, we have

$$(a)\quad \{(c^2)^3\}^7 = \{c^6\}^7 = c^{42};$$
$$(b)\quad \{(x^m)^n\}^p = \{x^{mn}\}^p = x^{mnp}.$$

w IV: ***The Power of a Product***

$$(xy)^m = x^m y^m.$$

The proof of this law is left as an exercise. State the law in ds.

w V: ***The Power of a Quotient***

$$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}.$$

The proof is left as an exercise. State the law in words.

mple 1. Simplify $\dfrac{(x^m)^2}{x^{m+n}} \cdot \dfrac{(x^n)^2}{x^{m-n}}$.

We have

$$\frac{(x^m)^2}{x^{m+n}} \cdot \frac{(x^n)^2}{x^{m-n}} = \frac{x^{2m}}{x^{m+n}} \cdot \frac{x^{2n}}{x^{m-n}} \qquad \text{(Law III)}$$
$$= \frac{x^{2m+2n}}{x^{2m}} \qquad \text{(Law I)}$$
$$= x^{2n}. \qquad \text{(Law II)}$$

mple 2. Use the laws of indices to simplify $\dfrac{27^{10}}{9^{14}}$.

At first sight one might think that the laws of indices do n apply because the powers have different bases. Note, hov ever, that $27^{10} = (3^3)^{10} = 3^{30}$,
and $9^{14} = (3^2)^{14} = 3^{28}$.

Since both powers can be written with the base 3, we have

$$\frac{27^{10}}{9^{14}} = \frac{3^{30}}{3^{28}} = 3^2 = 9.$$

EXERCISES

For oral answers

Simplify:

1. $x^2 \cdot x^3$.
2. $z^6 \div z^2$.
3. $(a^2)^3$.
4. $x^r \cdot x^s$.
5. $z^a \div z^b$.
6. $(a^c)^m$.
7. $a^2 \cdot a^4 \cdot a^6$.
8. $m^5 \cdot m^4 \div m^3$.
9. $(x^5)^4$.
10. $5x^m \cdot 2x^{2m}$.
11. $8y^{2m} \div 4y^m$.
12. $(x^a)^{2a}$.
13. $5^7 \cdot 5^4$.
14. $9^{13} \div 9^{11}$.
15. $(2^3)^3$.
16. $3^4 \cdot 3^{12} \div 3^{14}$.
17. $(3x)^3$.
18. $\left(\frac{x}{2}\right)^4$.
19. $(ab^2)^3$.
20. $(x^3)^4 \div x^7$.
21. $\{(y^3)^4\}^2$.
22. $\{(-1)^2\}^3$.
23. $\left(\frac{x^2}{y^3}\right)^3$.
24. $(m^3)^4 \div (m^2$
25. $(-2)^8 \div (-2)^6$.
26. $\frac{(m^2n)^4}{(mn^2)^2}$.
27. $x^m \cdot x^n \cdot x^k$.

For written answers

28. $a^{x-y} \cdot a^{x+y}$.
29. $m^a \cdot m^{a+b} \cdot m^{b-2a}$.
30. $\frac{a^{x+y}}{a^{x-y}}$.
31. $(p^2q^3r^4)^2$.
32. $x^{3m-2n} \cdot x^{m+2n}$.
33. $(a^{x-y})^{x+y}\ a^{x^2+}$
34. $125a^5b^x \div 25a^xb^4$.
35. $(x^2y^2z^2)^a \div (x^ay^az^a)^2$.
36. $\frac{(x^n)^3}{x^{2m+n}} \cdot \frac{(x^m)^3}{x^{m+2n}}$.
37. $\frac{(2^n)^5 \times (2^n)^{n-1}}{2^{n+3} \times 2^{n-3} \times 4^n}$.

38. $\dfrac{(3^a)^3 \times (3^{a+2})^{a-2}}{(3^a)^a}$.

39. $\left(\dfrac{x}{y}\right)^2 \cdot \left(\dfrac{y}{z}\right)^3 \cdot \left(\dfrac{z}{x}\right)^4$.

40. $\left(\dfrac{a}{b}\right)^x \left(\dfrac{b}{a}\right)^y (ab)^{x-y}$.

41. $\dfrac{m^{2a+b+c} \cdot m^{a+2b+2c}}{(m^3)^{a+b+c}}$.

42. Express (*a*) 8^4 as a power of 2; (*b*) 27^5 as a power of 3; (*c*) 125^3 as a power of 5; (*d*) a million as a power of 10.

43. Express (*a*) 5^{12} as a power of 125; (*b*) 3^{16} as a power of 9^2; (*c*) a^{8r} as a power of a^r; (*d*) a^{8r} as a power of a^{2r}.

44. Divide (*a*) 125^5 by 25^7; (*b*) 81^6 by 27^8.

45. Simplify $\dfrac{3^2 \times 9^3 \times 27^4}{81^4}$.

46. Simplify $\dfrac{2^n \times 4^{n-1} \times 16^{n+2}}{(8^2)^{n+1}}$.

Some New Definitions

The definition of x^m on page 127 is valid as long as m is a itive integer; if m is not a positive integer, the definition is iout meaning. What significance, then, are we to attach to following:

$$x^{\frac{1}{2}},\ x^{\frac{1}{4}},\ y^{\frac{3}{4}},\ a^{-\frac{2}{3}},\ x^0,\ y^{-2},\ \text{and so on?}$$

e answer is this: at present, they have no meanings since the ces are not positive integers; however, we are free to assign anings to such expressions by means of appropriate definitions.

In defining these powers with fractional or negative indices, natural for us to seek definitions that lead to useful results that preserve the characteristics of the positive, integral ces with which we are already familiar. In particular, the

$$x^m \cdot x^n = x^{m+n}$$

was proved when m and n are positive integers. We desire t preserve this law for all values of m and n. We shall therefo make our definitions so that the results are in accordance wi this law of multiplication.

The Meaning of a Fractional Index. Consider $x^{\frac{1}{2}}$. If th power is to obey the index law of multiplication, we have

$$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x.$$

Thus, it appears that $x^{\frac{1}{2}}$, when multiplied by itself, equals We also have

$$\sqrt{x} \cdot \sqrt{x} = x.$$

These two equations suggest the following ***definition*** for $x^{\frac{1}{2}}$:

$$x^{\frac{1}{2}} = \sqrt{x}.$$

Similarly, by the law of multiplication, we get

$$x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x.$$

Hence we make the ***definition***:

$$x^{\frac{1}{3}} = \sqrt[3]{x}.$$

Proceeding in the same way, we see that we should define th

$$x^{\frac{1}{4}} = \sqrt[4]{x},$$

and, in general,

$$\boxed{x^{\frac{1}{n}} = \sqrt[n]{x}}$$

Examples. $25^{\frac{1}{2}} = \sqrt{25} = 5$; $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$; $256^{\frac{1}{4}} = \sqrt[4]{256} =$

From the law of powers, we have

$$\sqrt{x^3} = (x^3)^{\frac{1}{2}} = x^{\frac{3}{2}},$$

and also

$$x^{\frac{3}{2}} = (x^{\frac{1}{2}})^3 = (\sqrt{x})^3.$$

s suggests that the index 3/2 should indicate either
(1) the cube of the square root of the base, or
(2) the square root of the cube of the base.

ilarly, we have

$$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m},$$

$$x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m.$$

Definition:

$$\boxed{x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m}$$

Note that the index m/n indicates either ***the mth power of nth root of the base,*** or ***the nth root of the mth power he base.*** The order of operation is immaterial. In a given erical problem, it is advisable to take the root first, where ible.

mple 1. Find the value of $16^{\frac{3}{4}}$.

By definition, we have

$$16^{\frac{3}{4}} = (\sqrt[4]{16})^3,$$
$$16^{\frac{3}{4}} = \sqrt[4]{16^3}.$$

Since the fourth root of 16 is 2, the first equation leads more kly to a simple answer:

$$16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 2^3 = 8.$$

second equation calls for the taking of the 4th root of 16^3, the 4th root of 4096. One might fail to recognize that this ber is a perfect 4th power, and so miss the simple answer 8.

Example 2. Simplify $\sqrt{9^3} \cdot \sqrt[3]{27^2}$.

We have

$$\sqrt{9^3} \cdot \sqrt[3]{27^2} = 9^{\frac{3}{2}} \cdot 27^{\frac{2}{3}} = (3^2)^{\frac{3}{2}} \cdot (3^3)^{\frac{2}{3}} = 3^3 \cdot 3^2 = 2$$

The Meaning of a Zero Index. By the law of multiplicati we have

$$x^0 \cdot x^2 = x^{0+2} = x^2,$$

which gives

$$x^0 = \frac{x^2}{x^2} = 1.$$

Since we wish the power with the index zero to obey the sa law of multiplication as other powers, we should define x^0 have the value 1.

DEFINITION:

$$\boxed{x^0 = 1}$$

where x is any number except 0.

Examples. $5^0 = 1$; $(ab^2c)^0 = 1$; $(-276.3)^0 = 1$; $\{(xy)^0\}^{25} =$

The Meaning of a Negative Index. In order that the l of multiplication should hold for negative indices, we must h

$$a^{-2} \times a^2 = a^{-2+2} = a^0 = 1,$$

and

$$a^{-m} \times a^m = a^{-m+m} = a^0 = 1.$$

These equations give, respectively,

$$a^{-2} = \frac{1}{a^2} \quad \text{and} \quad a^{-m} = \frac{1}{a^m}.$$

DEFINITION:

$$\boxed{x^{-m} = \frac{1}{x^m}}$$

ollows that

$$x^m = \frac{1}{x^{-m}}$$

The value of a fraction is unchanged if a ***factor*** *is moved from the numerator to the denominator, or vice versa, provided that the sign of the index of the factor is changed.*

or convenience, we collect the definitions of powers with tional, zero, and negative indices.

$$x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$$
$$x^0 = 1$$
$$x^{-m} = \frac{1}{x^m}$$

mple 1. $5^{-2} = \frac{1}{5^2} = \frac{1}{25} = 0.04.$

mple 2. $81^{-\frac{3}{4}} = \frac{1}{81^{\frac{3}{4}}} = \frac{1}{(\sqrt[4]{81})^3} = \frac{1}{3^3} = \frac{1}{27}.$

mple 3. Simplify (*a*) $\frac{2^{-1} \cdot 3^{-2}}{4^{-2}}$; (*b*) $\frac{2^{-1} + 3^{-2}}{4^{-2}}.$

(*a*) Using the principle on page 134, we get

$$\frac{2^{-1} \cdot 3^{-2}}{4^{-2}} = \frac{4^2}{2^1 \cdot 3^2} = \frac{16}{18} = \frac{8}{9}.$$

(*b*) Why does the principle on page 134 not enable us to move 2^{-1} and 3^{-2} to the denominator as in (*a*)?

We have
$$\frac{2^{-1}+3^{-2}}{4^{-2}} = \frac{\frac{1}{2}+\frac{1}{3^2}}{\frac{1}{4^2}}$$
$$= \frac{\frac{1}{2}+\frac{1}{9}}{\frac{1}{16}}$$
$$= \frac{72+16}{9} \quad \text{(Why?)}$$
$$= \frac{88}{9} = 9\frac{7}{9}.$$

Example 4. Simplify $\left(\frac{25x^2y^{-2}}{9z^{-4}}\right)^{-\frac{3}{2}}$.

Using the rule of powers, we get
$$\left(\frac{25x^2y^{-2}}{9z^{-4}}\right)^{-\frac{3}{2}} = \frac{25^{-\frac{3}{2}}\cdot x^{-3}\cdot y^3}{9^{-\frac{3}{2}}\cdot z^6} = \frac{9^{\frac{3}{2}}\cdot y^3}{25^{\frac{3}{2}}\cdot x^3\cdot z^6} = \frac{27y}{125x}$$

EXERCISES

For oral answers

State the meaning of:

1. $x^{\frac{1}{2}}$.
2. $x^{-\frac{1}{2}}$.
3. $y^{\frac{1}{3}}$.
4. $y^{-\frac{1}{3}}$.
5. $m^{\frac{1}{4}}$.
6. $z^{\frac{1}{n}}$.
7. k^0.
8. x^{-1}.
9. a^{-5}.
10. z^{-a}.
11. $(-2)^0$.
12. $x^{\frac{3}{4}}$.
13. $a^{\frac{2}{3}}$.
14. $m^{\frac{a}{c}}$.
15. $(abc)^0$.

16. $c^{-\frac{3}{4}}$. 17. $x^{-\frac{3}{2}}$. 18. $n^{-\frac{s}{x}}$.

press the following as powers of the base x:

19. $\sqrt{x}$. 20. $\sqrt[3]{x}$. 21. $\sqrt{x^3}$.

22. $\sqrt[3]{x^2}$. 23. $(\sqrt[4]{x})^5$. 24. $\sqrt[2]{x^c}$.

25. $\dfrac{1}{\sqrt[3]{x}}$. 26. $\dfrac{1}{(\sqrt{x})^3}$. 27. $\dfrac{1}{\sqrt[n]{x^m}}$.

press the following with positive indices:

28. $3a^{-2}$. 29. x^2y^{-3}. 30. $\dfrac{m^{-n}}{5^{-2}}$.

31. $\dfrac{a^2}{cb^{-2}}$. 32. $\dfrac{1}{(3z)^{-2}}$. 33. $\dfrac{1}{3z^{-2}}$.

34. $\dfrac{x^{-2}y^2}{5^{-1}\,z^{-2}}$. 35. $\dfrac{4a^3b^{-3}}{3c^{-2}}$. 36. $x^{-2}+y^{-2}$.

press the following without denominators:

37. $\dfrac{2}{x^2}$. 38. $\dfrac{a^2b^2}{c^3}$. 39. $\dfrac{m^3n^3}{3r^3}$.

40. $\dfrac{x^4y^4}{z^{-5}}$. 41. $\dfrac{6a}{k^2s^{-3}}$. 42. $\dfrac{4}{c^3(a+b)^{-2}}$.

For written answers

d the value of each of the following:

43. $9^{\frac{1}{2}}$. 44. $9^{-\frac{1}{2}}$. 45. $25^{\frac{3}{2}}$.

46. $25^{-\frac{3}{2}}$. 47. $(1.03)^0$. 48. 2^{-2}.

49. $1000^{-\frac{1}{3}}$. 50. $(-8)^0$. 51. $16^{-\frac{3}{4}}$.

52. $(0.25)^{-\frac{1}{2}}$. 53. $(0.09)^{\frac{3}{2}}$. 54. $(\frac{3}{4})^{-3}$.

55. $(-5)^{-2}$. 56. $3^4 \cdot 3^{-3}$. 57. $(x^0)^{-\frac{3}{4}}$.

58. $81^{\frac{1}{4}}$. 59. $32^{-\frac{1}{5}}$. 60. $(\frac{1}{4})^{-\frac{5}{2}}$.

Simplify:

61. $(16^{-\frac{3}{4}})^0$. 62. $\dfrac{1}{16^{-\frac{3}{4}}}$. 63. $64^{1.5}$.

64. $(0.2)^{-2}$. 65. $(0.001)^{\frac{2}{3}}$. 66. $81^{1.25}$.

67. $\sqrt[4]{625^{-1}}$. 68. $\dfrac{1}{2^{-1} \cdot 3^{-1}}$. 69. $\dfrac{1}{2^{-1}+3^{-1}}$

70. $\left(\dfrac{100}{121}\right)^{-\frac{1}{2}}$. 71. $\left(\dfrac{40}{135}\right)^{-\frac{1}{3}}$. 72. $625^{0.75}$.

73. $\dfrac{5^{-2} \cdot 10^{-1}}{5^{-3}}$. 74. $\dfrac{5^{-2}+10^{-1}}{5^{-3}}$.

75. $\sqrt[4]{16^3} \times \sqrt{49^3}$. 76. $\sqrt[5]{2^3(2^4)^4 \div 2^4}$.

77. $\left(\dfrac{4x^4y^{-2}}{9z^{-6}}\right)^{\frac{1}{2}}$. 78. $\left(\dfrac{8x^3y^{-3}}{27a^{-6}}\right)^{-\frac{2}{3}}$.

79. $2^{-\frac{3}{4}} \div 4^{-\frac{2}{3}}$. 80. $\left\{(b^{-\frac{3}{4}})^{\frac{1}{3}}\right\}^{-2}$.

81. $27^{\frac{2}{3}} + 16^0 + (2^9)^{\frac{1}{3}} - \dfrac{2^{\frac{3}{5}}}{2^{-\frac{2}{5}}} + \left(\dfrac{4}{9}\right)^{\frac{1}{2}}$.

82. $3^2 - 7^0 - 9^{\frac{1}{2}} - \left(\dfrac{1}{8}\right)^{-\frac{1}{3}} + 0^4 + \left\{\left(\dfrac{1}{4}\right)^{\frac{3}{4}}\right\}^{-2}$.

Operations with Indices

The use of integral, fractional, negative and zero indices is based on the laws of operation on page 127. Until these laws are mastered, you will find difficulty in dealing with powers. The following examples illustrate some of the ways in which familiar problems may involve our new kinds of indices.

Note that a systematic arrangement of the work is important. a rule, in multiplication and division, *the work is facilitated by anging the expressions in descending powers.* For example,

$$5 + 3x - x^2 + x^{-1} + 3x^{-2}$$

uld be written

$$-x^2 + 3x + 5 + x^{-1} + 3x^{-2}.$$

mple 1. Multiply $2x^{\frac{1}{4}} - 1 - 3x^{-\frac{1}{4}}$ by $3x^{\frac{1}{4}} + 2 - 2x^{-\frac{1}{4}}$.

We have

$$\begin{array}{r}
2x^{\frac{1}{4}} - 1 - 3x^{-\frac{1}{4}} \\
3x^{\frac{1}{4}} + 2 - 2x^{-\frac{1}{4}} \\
\hline
6x^{\frac{1}{2}} - 3x^{\frac{1}{4}} - 9 \\
4x^{\frac{1}{4}} - 2 - 6x^{-\frac{1}{4}} \\
- 4 + 2x^{-\frac{1}{4}} + 6x^{-\frac{1}{2}} \\
\hline
6x^{\frac{1}{2}} + x^{\frac{1}{4}} - 15 - 4x^{-\frac{1}{4}} + 6x^{-\frac{1}{2}}
\end{array}$$

mple 2. Divide

(*a*) $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 6$ by $x^{\frac{1}{3}} - 3$;

(*b*) $3a^{3x} - 11a^{x} + 8a^{-x} + 4a^{-3x}$ by $3a^{2x} - 5 - 2a^{-2x}$.

$$\begin{array}{r}
x^{\frac{1}{3}} + 2 \\
\hline
(a)\ x^{\frac{1}{3}} - 3 \overline{)}\ x^{\frac{2}{3}} - x^{\frac{1}{3}} - 6 \\
x^{\frac{2}{3}} - 3x^{\frac{1}{3}} \\
\hline
2x^{\frac{1}{3}} - 6 \\
2x^{\frac{1}{3}} - 6 \\
\hline
0
\end{array}$$

$$
\begin{array}{rl}
 & a^x - 2a^{-x} \\
\hline
(b)\ 3a^{2x} - 5 - 2a^{-2x}) & 3a^{3x} - 11a^x + 8a^{-x} + 4a^{-} \\
 & 3a^{3x} - 5a^x - 2a^{-x} \\
\hline
 & -6a^x + 10a^{-x} + 4a^{-} \\
 & -6a^x + 10a^{-x} + 4a^{-} \\
\hline
 & 0
\end{array}
$$

Example 3. Sometimes an operation can be easily perform by writing certain expressions as products in a manner simi to that used in factoring. Thus

$$x - y = (x^{\frac{1}{2}})^2 - (y^{\frac{1}{2}})^2 = (x^{\frac{1}{2}} - y^{\frac{1}{2}})\ (x^{\frac{1}{2}} + y^{\frac{1}{2}}).$$

If we wish to divide $x-y$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$, we may proceed as follo

$$\frac{x-y}{x^{\frac{1}{3}}-y^{\frac{1}{3}}} = \frac{(x^{\frac{1}{3}}-y^{\frac{1}{3}})\ (x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})}{x^{\frac{1}{3}}-y^{\frac{1}{3}}} = x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}.$$

NOTE: This method of writing certain expressions as produ is *not* factoring according to the definition on page 79. If were to call the process factoring, then we should have to adr that $x-y$ has an unlimited number of factors, since

$$x^{\frac{1}{2}} - y^{\frac{1}{2}} = (x^{\frac{1}{4}} - y^{\frac{1}{4}})\ (x^{\frac{1}{4}} + y^{\frac{1}{4}}), \text{ and so on.}$$

REVIEW EXERCISES ON CHAPTER VI

A

1. Multiply (*a*) x^2, x^3, and $2x^5$; (*b*) $1.5a^{\frac{3}{4}}$, $4a^{\frac{1}{2}}$, and (*c*) $7m^{-4}$, $m^{2.5}$, and $-m^{\frac{3}{2}}$.

2. Divide (*a*) $6k^6$ by $2k^3$; (*b*) $9x^{\frac{4}{5}}$ by $3x^{\frac{3}{5}}$; (*c*) $4.5a^{-2}$ by $3a$

3. Simplify (*a*) $\left\{(y^2)^4\right\}^{\frac{1}{2}}$; (*b*) $(x^{\frac{3}{4}})^{-2}$; (*c*) $\left\{(a^{-3})^{\frac{1}{4}}\right\}^{-2}$.

4. Write the square root of (*a*) $4x^{-2}$, (*b*) $9a^{\frac{4}{3}}$, (*c*) $25m^{-}$

5. Find the value of (*a*) $16^{\frac{1}{2}}$, (*b*) $16^{-\frac{1}{2}}$, (*c*) $\left(\frac{1}{16}\right)^{-\frac{1}{2}}$, (*d*) $16^{-\frac{3}{4}}$.

6. Find the value of $8^0 + 32^{\frac{3}{5}} - (27^2)^{\frac{1}{3}} - 0^3 + \left(\frac{1}{4}\right)^{-\frac{1}{2}}$.

7. Multiply $3x^{\frac{1}{2}} - 5$ by $2x^{\frac{1}{2}} + 4$.

8. Multiply $5x^{\frac{2}{3}} + 3$ by $5x^{\frac{2}{3}} - 3$.

9. Divide $2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 2$ by $x^{\frac{1}{3}} + 2$.

10. Divide $5a^{-4} + 14 - 3a^4$ by $5a^{-2} - a^2$.

11. Write the square of (*a*) $x^{\frac{1}{2}} + y^{\frac{1}{2}}$, (*b*) $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

B

12. Simplify $(x^{\frac{1}{5}} \div x^{\frac{3}{10}})^{-5} \cdot \sqrt{x}$.

13. Simplify $\dfrac{6x - \sqrt{x} - 12}{9x - 16}$.

14. Multiply $2m^{\frac{1}{3}} - 1 - 3m^{-\frac{1}{3}}$ by $3m^{\frac{1}{3}} + 2 - 2m^{-\frac{1}{3}}$.

15. Simplify $\left(\sqrt{\frac{1}{16}}\right)^{-1} + \left(\frac{1}{25}\right)^{-\frac{1}{2}} - \left(3\frac{3}{8}\right)^{\frac{2}{3}}$.

16. Divide $21x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$ by $3x^{\frac{1}{3}} + 1$.

17. Multiply $c + 5c^{\frac{3}{4}} + 6c^{\frac{1}{2}}$ by $c^{\frac{1}{4}} - 1 - c^{-\frac{1}{4}}$.

18. Divide $25x^{\frac{1}{2}} - 9y^{\frac{1}{2}}$ by $5x^{\frac{1}{4}} + 3y^{\frac{1}{4}}$.

19. Divide the product of $x - 5x^{\frac{1}{2}} + 6$ and $6x - x^{\frac{1}{2}} - 2$ by $-11x^{\frac{1}{2}} + 6$.

20. Multiply $3a^{-\frac{1}{3}} + a + 2a^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - 2$.

21. Simplify (*a*) $\dfrac{2^{-1}}{4^{-\frac{2}{3}}}$; (*b*) $\left\{(4^{-\frac{4}{3}})^{\frac{1}{4}}\right\}^{\frac{3}{2}}$.

22. Divide $8x - y$ by $4x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.

23. Divide $x^{4a} - y^{4a}$ by $x^a - y^a$.

24. Divide $9a^{\frac{4}{5}} - 9a^{\frac{2}{5}} - 25 + 23a^{-\frac{2}{5}} + 6a^{-\frac{4}{5}}$ by $3a^{\frac{3}{5}} - 4a^{\frac{1}{5}} - a^{-\frac{1}{5}}$.

25. Multiply $8a^{-2} + 7a^{-1} + 6$ by $2a^{-1} - 1$.

26. Multiply $a^{\frac{3}{4}}b^{-\frac{1}{2}} + 1 + a^{-\frac{1}{4}}b^{\frac{1}{4}}$ by $a^{\frac{3}{4}}b^{-\frac{1}{2}} + 1 - a^{-\frac{1}{4}}b^{\frac{1}{4}}$.

27. Find the square of $3x^{\frac{3}{2}}y^{\frac{1}{2}} + 2 - x^{-\frac{3}{2}}y^{-\frac{1}{2}}$.

28. Simplify $\dfrac{x^{\frac{4}{3}}y^{\frac{1}{2}}}{z^{-\frac{1}{3}}} \cdot \dfrac{x^{-\frac{1}{3}}z^{\frac{2}{3}}}{y^{-\frac{1}{3}}} \div \dfrac{x^{-\frac{1}{2}}y^{-\frac{1}{6}}}{z}$.

29. If $a=0$, $b=2$, and $c=-2$, find the numerical value of $3a^ba^a + 2^bb^cc^a - (5^bc^c)^a$.

30. Divide $m^{\frac{4}{3}} + m^{\frac{2}{3}}n^{\frac{2}{3}} + n^{\frac{4}{3}}$ by $m^{\frac{2}{3}} - m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}}$.

31. Simplify $\dfrac{3^n \times (3^{n-1})^n}{3^{n+1} \times 3^{n-1} \times 9^{-n}}$.

32. Divide $16x^{-3} - 6x^{-2} + 5x^{-1} + 6$ by $1 + 2x^{-1}$.

33. Divide $x^{\frac{4}{3}} - 4x^{\frac{2}{3}} + 3 - 6x^{-\frac{1}{3}}$ by $3x^{-\frac{1}{3}} + x + 2x^{\frac{2}{3}}$.

34. Find the value of $\sqrt[3]{27^2} \times \sqrt[5]{32^3} \div (\sqrt[4]{16})^5$.

35. Simplify $7^{\frac{3}{4}} \times 7^{\frac{1}{2}} \times 7^{\frac{1}{4}} \times 49^{\frac{1}{4}} \times 5^{\frac{3}{5}} \times 25^{\frac{1}{5}}$.

36. Simplify $\dfrac{81^{\frac{3}{4}} - 81^{\frac{1}{4}} - 81^{\frac{1}{2}}}{\sqrt[3]{125^2} + \sqrt[3]{125}}$.

37. Simplify $(\sqrt[3]{4})^4 \times 5^{\frac{1}{3}} \times (\sqrt[3]{20})^2$.

38. Simplify $\dfrac{(a-5\sqrt{a}-14)\ (10a-29\sqrt{a}+10)}{(2a-19\sqrt{a}+35)}$.

39. Solve (*a*) $x^{\frac{3}{2}} = 27$; (*b*) $x^{\frac{4}{3}} = 16$; (*c*) $x^{\frac{5}{2}} = 32$.

40. Multiply $3x^{2a} - x^a + 5$ by $3x^{2a} + x^a - 5$.

41. Find the square of $3a^{\frac{3}{2}}b^{\frac{1}{2}} - 5ab - 2a^{\frac{1}{2}}b^{\frac{3}{2}}$.

C

42. If the divisor is $5x^{\frac{2}{3}} + 4$, the quotient $3x^{\frac{1}{3}} - 3x^{-\frac{1}{3}} + 2x^{-1}$, the remainder $2x^{-1}$, find the dividend.

43. Divide $x + 2x^{\frac{5}{6}} - 2x^{\frac{1}{6}} + 2x^{-\frac{1}{6}} - 2x^{-\frac{5}{6}} - x^{-1}$ by $x^{\frac{1}{3}} - x^{-\frac{1}{3}}$.

44. Divide $(\sqrt[3]{x})^2 + 2\sqrt[3]{x} - 16x^{-\frac{2}{3}} - \dfrac{32}{x}$ by $(\sqrt{x})^{\frac{1}{3}} + 4x^{-\frac{1}{6}} - \dfrac{4}{\sqrt{x}}$.

45. Divide $4a^{\frac{2}{3}} - 8\sqrt[3]{a} - 5 + 10a^{-\frac{1}{3}} + \dfrac{3}{\sqrt[3]{a^2}}$ by $2(\sqrt[3]{a})^{\frac{5}{4}} - \sqrt[3]{a})^{\frac{1}{4}} - 3a^{-\frac{1}{4}}$.

46. Multiply $2x^3 - \sqrt[3]{x} - 3x^{-1}$ by $2x - \dfrac{3}{\sqrt[3]{x}} - \dfrac{1}{(\sqrt[3]{x})^5}$.

47. If $2m = e^x + e^{-x}$ and $2n = e^x - e^{-x}$, find $m^2 + n^2$.

48. Multiply $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$ by $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}} - {}^{\frac{1}{3}}a^{\frac{1}{3}}$.

49. Divide the product of $4m^2 - 5m - 4 - 7m^{-1} + 6m^{-2}$ $3m - 4 + 2m^{-1}$ by $3m - 10 + 10m^{-1} - 4m^{-2}$.

50. Simplify $\dfrac{5(3^n) - 4(3^{n-1}) + 3^{n-2}}{2(3^n) + 3^{n-1} - 4(3^{n-2})}$.

51. Show that the following three numbers measure the s of a right triangle:

$a = 2^{4n} + 2^{2n+1}$; $\quad b = 2^{4n} - 2^{4n-2} - 2^{2n} - 1$;

$c = 2^{4n} + 2^{4n-2} + 2^{2n} + 1$.

CHAPTER VII

EQUATIONS AND PROBLEMS

Equations and Their Solution

In Chapter I your attention was called to the basic princip for solving equations (page 8). It is necessary now to impr your skill in the use of these principles by solving equations more varied types.

Example. Solve:

$$\tfrac{2}{3}(8x - 11) - \tfrac{5}{9}(14x - 23) = \frac{6x + 9}{4} - 6\tfrac{2}{3}.$$

The coefficients in this equation involve numerical fr tions. These are got rid of by multiplying both sides of equation by a suitable number. This number must b multiple of 3, of 9, and of 4. The number 36 would ser also 72 or any other multiple of 36. The smallest multip which will get rid of the fractions is 36, the ***least comm denominator.***

If $\tfrac{2}{3}(8x - 11) - \tfrac{5}{9}(14x - 23) = \frac{6x + 9}{4} - 6\tfrac{2}{3},$

then $24(8x - 11) - 20(14x - 23) = 9(6x + 9) - 20 \times$

Hence $192x - 264 - 280x + 460 = 54x + 81 - 240.$

Therefore $192x - 280x - 54x = 81 - 240 + 264 -$

or $-142x = -355,$

from which $x = \frac{-355}{-142} = \frac{5}{2}.$

Verification: If $x = 5/2$, then the left side equals

$\tfrac{2}{3}(20-11) - \tfrac{5}{9}(35-23) = \tfrac{2}{3}\times 9 - \tfrac{5}{9}\times 12 = 6 - 6\tfrac{2}{3} = -$

and the right side equals $\frac{15 + 9}{4} - 6\tfrac{2}{3} = 6 - 6\tfrac{2}{3} = -$

EXERCISES

ve:

1. $3(2x - 8) = 4(5 - 2x)$.

2. $2 + x/3 = 5 - x/2$.

3. $\dfrac{5x - 7}{6} = \dfrac{4x + 7}{9}$.

4. $\frac{5}{6}x - \frac{3}{4} = \frac{2}{3}x - \frac{1}{2}$.

5. $\frac{3}{7}y - \frac{1}{5} = \frac{2}{5}y - \frac{1}{7}$.

6. $4.5(2.4 - 0.2y) = 8.55$.

7. $0.08(x - 0.5) = 1.2(3 - 0.1x)$.

8. $0.4(9 - 1.2x) = 5(0.06x - 2.4)$.

9. $3(2x + 21) - 5(3x - 16) = 71 - 6(68 - x)$.

10. $\dfrac{2x - 1}{4} + \dfrac{8x + 3}{20} = \dfrac{4x + 2}{15} + \dfrac{3x + 1}{5}$.

11. $\dfrac{x + 1}{12} - \dfrac{11x + 1}{84} = \dfrac{x - 5}{20} - \dfrac{2x - 7}{21}$.

12. $\frac{4}{5}(2x - 7) - \frac{1}{6}(3x + 5) = \frac{3}{4}(x + 9) - 9\frac{1}{3}$.

13. $\frac{3}{5}(2x - 5) + \dfrac{x}{4} = \frac{1}{2}(\frac{3}{2} - x) - \frac{48}{5}$.

14. $\frac{2}{5}(5 - 2x) - \frac{1}{6}(1 + \dfrac{x}{5}) = 2 - \dfrac{x}{3}$.

15. $\frac{2}{3}(x - 1) - \frac{2}{5}(4x - 5) = 3 - \dfrac{7x + 5}{10}$.

Literal equations. We consider now some equations which olve letters other than the unknown. These letters represent wn numbers although their numerical values have not been cified. In most equations the unknown is represented by a er near the end of the alphabet while the other letters in the ation represent known numbers. For the problems which ow no new principles are necessary; the basic principles ve-mentioned are sufficient for their solution.

Example 1. Solve the equation $V = \frac{1}{2}h(a + b)$ for h, for a, a
for b.

In this formula it is possible to regard any one of
letters as the unknown. We have

$$2V = h(a + b).$$

Hence $$h = \frac{2V}{a + b}.$$

Also $$a + b = \frac{2V}{h},$$

from which $$a = \frac{2V}{h} - b \quad \text{and} \quad b = \frac{2V}{h} - a.$$

Example 2. Solve for x: $$\frac{ax - b}{c} = \frac{cx + d}{a}.$$

If this equation is true, then

$$a(ax - b) = c(cx + d).$$

Hence $$a^2x - ab = c^2x + cd,$$

and $$a^2x - c^2x = ab + cd,$$

or $$(a^2 - c^2)x = ab + cd,$$

whence $$x = \frac{ab + cd}{a^2 - c^2}.$$

Verification: If $x = \frac{ab + cd}{a^2 - c^2}$, the left side of the given eq
tion is

$$\frac{a\left(\frac{ab + cd}{a^2 - c^2}\right) - b}{c} = \frac{a^2b + acd - a^2b + bc^2}{c(a^2 - c^2)} = \frac{c(ad + bc)}{c(a^2 - c^2)} = \frac{ad + b}{a^2 - c}$$

right side is

$$\frac{c\left(\frac{ab+cd}{a^2-c^2}\right)+d}{a} = \frac{abc+c^2d+a^2d-c^2d}{a(a^2-c^2)} = \frac{a(bc+ad)}{a(a^2-c^2)} = \frac{ad+bc}{a^2-c^2}.$$

EXERCISES

Solve for x:

1. $ax + b = c$.
2. $3a - 5x = 7b$.
3. $mx + n = hx + k$.
4. $r(x - a) = s(x - b)$.
5. $3mx + 5n = 5nx + 3m$.
6. $(p + 1)x + 1 = p^2$.
7. $3mnx - 6m^2 = 5n^2x - 10mn$.
8. $ax + 3ab = bx + a^2 + 2b^2$.
9. $x/a + b = x/b + a$.
10. $cx/d + dx/c = c^2 + d^2$.

11. $\dfrac{x+1}{p+q} = \dfrac{x-1}{p-q}$.

12. $ab\left(\dfrac{x}{a} - \dfrac{1}{b}\right) + bc\left(\dfrac{x}{b} - \dfrac{1}{c}\right) + ca\left(\dfrac{x}{c} - \dfrac{1}{a}\right) = 0.$

13. $\dfrac{x-a}{ab} + \dfrac{x-b}{bc} + \dfrac{x-c}{ca} = 0.$

14. $(a - 1)(a - bx) = (b - 1)(b - ax)$.
15. $(a - b)(x - c) - (b - c)(x - a) = (c - a)(x - b)$.

16. $\dfrac{a-b}{2a}x + \dfrac{2a}{a-b} = \dfrac{a+b}{b}x + \dfrac{b}{a+b}$.

17. Solve the equation $C = E/R$ for E and for R.

18. Solve $I = \dfrac{M}{C} \times 100$ for M.

19. Solve $l = ar^{n-1}$ for a.
20. Solve $s = \frac{1}{2}n(a + l)$ for n, for a, and for l.
21. Solve $E = \frac{1}{2}mv^2$ for m.

22. Solve $\dfrac{a - rl}{1 - r} = s$ for r, for l, and for a.

23. Solve $\frac{1}{6}h(B_1 + B_2 + 4M) = V$ for h, for B_1, and for
24. Solve $s = ut + \frac{1}{2}at^2$ for u and for a.
25. Solve $v^2 = u^2 + 2as$ for a and for s.

More equations involving fractions. Some equations invo fractions in which the unknown occurs in denominators as well in numerators. Not all of such equations can be solved by present methods. The following examples reduce to sim equations. It is left for you to supply the reasoning on wh the various steps are based.

Example 1. Solve: $\dfrac{x}{x+1} - \dfrac{7}{12} = \dfrac{3x+1}{4x+4}$.

The least common denominator is $12(x + 1)$. When b sides of the equation are multiplied by $12(x + 1)$, we obt

$$12x - 7(x + 1) = 3(3x + 1),$$

or $$5x - 7 = 9x + 3,$$

whence $$-4x = 10,$$

and $$x = 10/-4 = -5/2.$$

Verify the root by substitution.

Example 2. Solve: $\dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{1}{x+2} - \dfrac{1}{x+3}$.

The least common denominator is the product of all denominators. If we look closely at the fractions, however observe that some work will be saved by combining fractions in pairs. By combining the first two and the two, we get

$$\frac{x + 1 - x}{x^2 + x} = \frac{x + 3 - x - 2}{x^2 + 5x + 6},$$

or
$$\frac{1}{x^2+x} = \frac{1}{x^2+5x+6}.$$

By multiplying both sides by the common denominator, we get
$$x^2 + 5x + 6 = x^2 + x,$$
or
$$4x = -6,$$
from which
$$x = -3/2.$$
Verify the root by substitution.

Two conventional terms. In the solution of Example 1, we had at one e the equation $5x - 7 = 9x + 3.$
subtracting $9x$ from each side and adding 7 to each side, we obtain
$$5x - 9x = 3 + 7.$$
effect of these steps is to move each of the terms $9x$ and -7 to the opposite of the equation and change its sign; the terms are said to be ***transposed*** the operation is called ***transposition.*** These names, however, are in ense necessary and, since they do not suggest the reasoning which justifies n, it is better to avoid them in favour of the basic principles of addition subtraction.

In Example 2, consider the effect of multiplying both sides of the equation
$$\frac{1}{x^2+x} = \frac{1}{x^2+5x+6}$$
he common denominator. The same result is obtained by ***cross-tiplication*** as explained on page 57. As this term does not suggest its explanation and as the process is liable to be misused, it is better to don the term and use the basic principle already given.

EXERCISES

ve:

1. $\frac{6}{x} = \frac{3}{4}.$

2. $\frac{4}{x+2} = \frac{2}{7}.$

3. $\frac{5}{3x+4} = \frac{3}{7}.$

4. $\frac{3}{5+x} = \frac{2}{4-x}.$

5. $\frac{3}{x} + 8 = \frac{5}{x} - 6.$

6. $\frac{6}{2x+5} - \frac{8}{4x-3} = 0.$

7. $\dfrac{x+1}{2x-3} = \dfrac{x-1}{2x+5}$.

8. $\dfrac{1}{x} - \dfrac{1}{a} = \dfrac{1}{b} - \dfrac{1}{x}$.

9. $a + \dfrac{b}{x} = c + \dfrac{d}{x}$.

10. $\dfrac{a}{x+1} = \dfrac{b}{x-1}$.

11. $\dfrac{m}{x-a} = \dfrac{n}{x+b}$.

12. $\dfrac{2x+3}{3x-5} = \dfrac{4x+5}{6x-7}$.

13. $\dfrac{1/x+5}{3/x+4} = \dfrac{2/x-6}{6/x-7}$.

14. $\dfrac{x+1}{2x} = \dfrac{2x-1}{4x} + \dfrac{3}{8}$.

15. $\dfrac{x+1}{2x+1} + \dfrac{7}{15} = \dfrac{3x-4}{6x+3}$.

16. $\dfrac{1}{x-3} - \dfrac{1}{x-1} = \dfrac{1}{x+1} - \dfrac{1}{x+3}$.

17. $\dfrac{3}{2x+2} + \dfrac{2}{3x-3} = \dfrac{x+5}{x^2-1}$.

18. $14 - \dfrac{56}{x+5} = 5 + \dfrac{18x}{2x+7}$.

19. $(3 - 6/x)(5 + 2/x) = (2 - 3/x)(9 + 4/x)$.

20. $\dfrac{5}{4x+4} - \dfrac{3}{2x+6} = \dfrac{2x-18}{x^2+4x+3}$.

21. $\dfrac{1}{2x-3} - \dfrac{1}{2x+3} = \dfrac{1}{2x+5} - \dfrac{1}{2x+11}$.

22. $\dfrac{3}{x+1} + \dfrac{5}{2x-2} = \dfrac{7}{3x+3} + \dfrac{2}{x-1}$.

23. $\dfrac{4x}{3x-5} = \dfrac{5}{8} + \dfrac{7x+5}{12x-20}$.

24. $\dfrac{x+1}{x^2-x} + \dfrac{x}{x^2-1} - \dfrac{2x-2}{x^2+x} = 0$.

25. $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{2}{x-c}$.

A fifth basic principle. The product $5 \times 0 = 0$; likewise $00{,}000 \times 0 = 0$ and $a \times 0 = 0$, no matter what number a is. neither a nor b is 0, then the product ab is not 0. It follows that *he product of two numbers is* 0, *then at least one of the numbers is* 0. is result furnishes us with a fifth basic principle in solving iations.

> ***Basic Principle***
>
> *If* $\boldsymbol{ab} = 0$, *then either* $\boldsymbol{a} = 0$ *or* $\boldsymbol{b} = 0$.

tice that the equation $ab = 0$ is satisfied if $a = 0$, whatever the value of b, and it is satisfied by $b = 0$, whatever be the ue of a.

This principle reminds us that in the game of algebra the yer ***zero*** will take a good deal of watching. He will cause you ch confusion unless you play strictly according to the rules.

Consider now an equation with an unknown, x.

If	$x(x-2) = 0$,
then	*either* $x = 0$ *or* $x - 2 = 0$.
Hence	*either* $x = 0$ *or* $x = 2$.

Notice that if $x(x-2) = 0$ then $x = 0$ *or* $x = 2$; no other number satisfy the equation. Conversely, $x = 0$ does satisfy the equation and 2 satisfies. Hence the equation has two roots 0 *and* 2, and these are only roots.

The following examples illustrate further the use of the basic principle.

If	$(x-3)(x+5) = 0$,
then	*either* $x = 3$ *or* $x = -5$.

Moreover, each of these numbers is seen to satisfy the equation. Hence the roots of this equation are 3 *and* -5.

If $(2x - 7)(3x + 4) = 0,$

then *either* $2x - 7 = 0$ *or* $3x + 4 = 0.$

That is, *either* $x = \frac{7}{2}$ *or* $x = -\frac{4}{3}$. Since each of these numb is found to satisfy the equation, the roots of the equation a 7/2 *and* −4/3.

Similarly, the roots of

$$(x - a)(x - b)(x - c) = 0$$

are a, b, and c.

To solve an equation means to find all its roots.

EXERCISES

Solve:

1. $(x - 4)(x + 9) = 0.$
2. $(3x - 6)(2x + 8) = 0.$
3. $(5x + 2)(2x + 5) = 0.$
4. $(3x - 1)(11x + 10) =$
5. $(x + 1)(x + 2)(x + 3) = 0.$
6. $(2x - 1)(3x - 2)(4x - 3) = 0.$
7. $(x + a)(x - b) = 0.$
8. $(ax - b)(cx + d) = 0.$
9. $\left(\frac{x}{2} - \frac{1}{3}\right)\left(\frac{x}{4} + \frac{1}{5}\right) = 0.$
10. $(0.1x - 1)(0.2x - 3) = 0.$
11. $(2x - 3.6)(5x - 7.5) = 0.$
12. $(x - p)(2x - q)(x + r)(2x + s) = 0.$

Quadratic equations. The equation $(x - 2)(x + 3) =$ seen to have the roots 2 and −3. If the factors on the left multiplied out, we have

$$x^2 + x - 6 = 0.$$

Since the meaning of the equation is not altered by the m tiplication, this equation has the roots 2 and −3. Moreover, t way of getting the second equation suggests that, if it were gi to begin with, the first step in solving it would be to factor left member. Now take some other examples.

ample 1. Solve: $4x^2 - 25 = 0.$

By factoring the left member we get

$$(2x - 5)(2x + 5) = 0$$

from which $x = 5/2$ or $-5/2.$

Verification: By substituting each answer in the left member of the original equation, we find that

$$4(5/2)^2 - 25 = 0 \quad \text{and that} \quad 4(-5/2)^2 - 25 = 0.$$

mple 2. Solve: $x^2 - 6x + 9 = 0.$

In factored form this equation is

$$(x - 3)^2 = 0.$$

Hence the only number that will satisfy the equation when substituted for x is 3. In order to remind ourselves that there are two factors on the left of this equation, we sometimes say that the equation has two roots 3 and 3, in other words, two equal roots 3, 3.

mple 3. Solve: $2x^2 - x - 21 = 0.$

When the left member of this equation is factored, we get

$$(2x - 7)(x + 3) = 0.$$

Hence the roots are $7/2$ and $-3.$

The equations in the three foregoing examples all involve the are of the unknown but no higher power. Equations of this d are called ***second-degree equations*** or ***quadratic equations.*** They play a large part in the applications of algebra.

EXERCISES

ve for x:

1. $x^2 - 9 = 0.$
2. $6x^2 - 24 = 0.$
3. $25x^2 - 81 = 0.$
4. $x^2 + 14x + 49 = 0.$
5. $x^2 - 5x + 6 = 0.$
6. $x^2 + 5x - 24 = 0.$

A DIGRESSION ON ZERO

Let x and y be two numbers different from zero and suppose that $x = y$ . . . (1)

Then $x^2 = xy$ . . . (2) (Multiply eac side of (1) by x

Hence $x^2 - y^2 = xy - y^2$. (3) (Subtract y^2)

and $(x - y)(x + y) = y(x - y)$. (4)

from which $x + y = y$ . . . (5) (Divide out $x - y$)

Since $x = y$, this gives $2y = y$ and hence $2 = 1$.

Again, by subtracting y from each side of (5), we get $x = 0$, which contradicts our original assumption that $x \neq 0$.

We have arrived at two absurd results. At what point did the fallac enter our reasoning? . . . Examine step (4). Since $x = y$ by (1 $x - y = 0$. Therefore (5), which is incorrect, was obtained from (4), whic is correct, by dividing each side by 0. This example shows that *division b zero is not permissible.* It has been said that, for students of mathematic the eleventh commandment is

"Thou shalt not divide by zero."

Now expose the fallacy in the following reasoning:

Let a and b be two numbers each greater than 0 and let c be the average.

Then $(a + b)/2 = c$ or $a + b = 2c$ . . . (1)

Hence $(a + b)(b - 2c) = 2c(b - 2c)$

or $ab - 2ac + b^2 - 2bc = 2bc - 4c^2$ . . (2)

Therefore $ab + b^2 - 2bc = 2ac + 2bc - 4c^2$ (3) (Add $2ac$ each side)

This gives $b(a + b - 2c) = 2c(a + b - 2c)$

from which $b = 2c$ . . . (4) (Divide o $a + b - 2$

Hence $2c$ is equal to b by (4) and is greater than b by (1).

7. $2x^2 + x - 6 = 0$.
8. $3x^2 - 5x - 12 = 0$.
9. $7x^2 - 30x + 8 = 0$.
10. $9x^2 + 30x + 25 = 0$.
11. $1.6x^2 - 8.1 = 0$.
12. $90x^2 - 181x + 90 = 0$.
13. $12x^2 + 8ax - 15a^2 = 0$.
14. $20x^2 + 61rx + 36r^2 = 0$.
15. $2k^2x^2 - 7kx - 15 = 0$.
16. $x^2 - 3x - kx + 3k = 0$.

A use for square roots. The equation $x^2 - 4 = 0$ can be ved by factoring the left member:

$$(x - 2)(x + 2) = 0,$$

ence $x = 2$ or -2.

We may adopt a different point of view in the solution. Let vrite the equation in the form

$$x^2 = 4.$$

ollows that x is a number whose square is 4. There are two n numbers, 2 and -2, which are called the square roots of 4. ese two square roots are written

$$\sqrt{4} = 2 \qquad \text{and} \qquad -\sqrt{4} = -2.$$

h numbers are often included in the composite symbol

$$\pm\sqrt{4} = \pm 2.$$

In a similar manner the quadratic equation

$$x^2 - 289 = 0$$

$$x^2 = 289$$

the two roots $x = \pm\sqrt{289} = \pm 17$.

We seem here to have a new principle in the solution of equations, namely if two positive numbers are equal a square root of one is equal to a square of the other. However, this fact has already been deduced from the fifth principle, so that no new principle is required.

From an equation like $x^2 = 4$ we could, by taking square roots, deduce $\pm x = \pm 2$. This stands for the four separate equations $x = 2$, $x = -2$,

$-x = 2$, and $-x = -2$. However, the first and fourth of these equati are seen to be equivalent; also the second and third are equivalent. He all the possible results are obtained by writing $x = \pm 2$.

A number like 289 which is the square of an integer is calle ***perfect square.*** The perfect square integers, in order of m nitude, are 0, 1, 4, 9, 16, 25, etc. Likewise a fraction such as 4 or 1.44 is called a perfect square if it is the square of anoth fraction. If a number is not a perfect square, then its square r can be expressed only approximately as a decimal fraction. T problem of finding such approximations will be considered in t following chapter.

Example. Solve: $5(2x - 1)^2 = 80.$

If the equation is true, then

$$(2x - 1)^2 = 16$$

from which $$2x - 1 = \pm 4.$$

If $2x - 1 = 4$, then $2x = 5$ and $x = 5/2$.
If $2x - 1 = -4$, then $2x = -3$ and $x = -3/2$.

Verification: If $x = 5/2$, the left side of the given equat equals $5(5 - 1)^2 = 5 \times 16 = 80$. If $x = -3/2$, the l side $= 5(-3 - 1)^2 = 5 \times 16 = 80$.

Hence the two roots of the equation are $5/2$ and $-3/2$.

EXERCISES

1. Write the squares of the integers from 1 to 20.
2. Find the values of $\sqrt{1.44}$, $\sqrt{0.81}$, $\sqrt{1\frac{11}{25}}$, $\sqrt{1\frac{48}{121}}$, $\sqrt{2}$

Solve by the method of square roots:

3. $x^2 = 225.$
4. $4x^2 - 9 = 0.$
5. $(5x - 2)^2 = 0.$
6. $(x - 1)^2 - 36 = 0.$
7. $(x + 10)^2 = 100.$
8. $(2x - 3)^2 = (x + 4)^2.$
9. $3(x + 2)^2 = 27.$
10. $16(2x - 3)^2 = 49.$

1. $(x + a)^2 = b^2$.
3. $9(x + m)^2 = 4(x - n)^2$.
5. $(9x^2 - 41)^2 = 1600$.
12. $(ax + b)^2 = (px + q)^2$.
14. $(x^2 - 5)^2 = 16$.
16. $(x^2 - 13a^2)^2 = 144a^4$.

The Use of Equations in Solving Problems

Equations furnish an effective tool for the solution of prob-
ıs. The following subsections give an indication of the variety
problems that can be solved by this means.

Problems on integers and fractions.

ımple. The denominator of a fraction exceeds the numerator
6. If the numerator and denominator are each diminished by
he fraction reduces to 2/3. Find the fraction.

If the numerator is denoted by x, the fraction is $\frac{x}{x + 6}$.

When numerator and denominator are diminished by 5, the fraction becomes

$$\frac{x - 5}{x + 6 - 5} \quad \text{or} \quad \frac{x - 5}{x + 1}.$$

Hence we have $$\frac{x - 5}{x + 1} = \frac{2}{3},$$

from which $$3x - 15 = 2x + 2,$$

or $$x = 17.$$

The required fraction is therefore 17/23. This fraction is easily seen to satisfy the conditions of the problem.

EXERCISES

1. If 2 less than 3 times a number is equal to 5 more than
ce the number, find the number. $3x - 2 = 2x + 5$
2. What number, if increased by 2, yields 4 less than twice
number?

③ A fraction with a numerator of 5 is halved by increasi the denominator by 6. Find the fraction.

④ Find the number such that 1 less than three-quarters of is 1 more than five-sixths of it.

5. If five-eighths of a number is as much more than 1/5 seven-tenths of the number is more than 1/2, find the number.

⑥ The denominator of a fraction exceeds the numerator 10. If the numerator and denominator are each increased by the fraction reduces to 3/5. Find the fraction.

⑦ The denominator of a fraction exceeds the numerator 13. If the numerator is diminished by 2 and the denomina by 5, the new fraction reduces to 1/3. Find the fraction.

8. The denominator of a fraction exceeds the numerator 9. The fraction obtained by increasing the numerator by 2 equal to that obtained by diminishing the denominator by Find the fraction.

⑨ What number is 16 times its reciprocal?

(*Recall that the reciprocal of a number x is $1/x$. It follows t the reciprocal of $1/x$ is x.*)

⑩ The sum of a number and its reciprocal is $2\frac{1}{30}$. Find t number.

11. Two numbers differ by 6 and their product is 112. Fi them.

12. Find the number whose square is 3 more than twice number.

⑬ The square of 4 more than a certain number is 34 m than the number. Find the number.

14. The denominator of a fraction exceeds its numerator by A second fraction is formed from the first by diminishing numerator by 2 and its denominator by 10. If the sum of the t fractions is 2, find the first fraction.

⑮ The denominator of a fraction exceeds its numerator b and a second fraction is formed from it by subtracting 4 fr each of its terms. When 5 times the first fraction is added to reciprocal of the second fraction, the sum is 6. Find the fi fraction.

Problems involving percentage.

mple. A dealer who sold radios at an advance of 20% on the t price found that when the cost of each radio was raised $2 same selling price represented an advance of 15% on the cost. d the original cost.

Let the cost of a radio be $\$x$.
Then the selling price is 120% of $\$x$ or $1.2x$ dollars.
The increased cost price is $\$(x + 2)$ and the selling price is
115% of $\$(x + 2)$ or $1.15(x + 2)$ dollars.
Since the selling price does not change, we have

$$1.2x = 1.15(x + 2),$$

from which $0.05x = 2.3$

and $$x = \frac{2.3}{0.05} = \frac{230}{5} = 46.$$

Hence the original cost of the radios was $46.

Verification: If the cost of the radios is $46, the selling price s $\$(1.2 \times 46) = \$55.20$. Also the selling price is $\$(1.15 \times 48) = \$55.20$.

EXERCISES

The first six may be answered orally

1. If 10% of a number is 42, what is the number?
2. A year's interest at 5% was $25. What was the principal?
3. An article cost $\$x$. What is the selling price if the article ld at (*a*) a gain of 20%, (*b*) a loss of 20%, (*c*) a gain of 25%, a gain of $33\frac{1}{3}\%$, (*e*) a loss of 40%, (*f*) a loss of 25%?
4. What is (*a*) 10% of $\$p$, (*b*) 25% of n days, (*c*) $33\frac{1}{3}\%$ of ns, (*d*) $p\%$ of $200, (*e*) $q\%$ of 60°, (*f*) $r\%$ of 75 marks, (*g*) $a\%$ *b*?
5. If 4% of a number is b, find the number.
6. If $a\%$ of a number is b, find the number.
7. The sum of two numbers is 60 and 10% of one is equal 5% of the other. Find the numbers.

8. (*a*) If the cost of an article is $\$5x$ and the gain is 20% the cost, what is the selling price? (*b*) If the cost is $\$4y$ and t loss is 25%, what is the selling price?

9. (*a*) If the cost of an article is $\$a$ and the gain is $b\%$ of t cost, find the selling price.

(*b*) If the cost is $\$a$ and the gain is $b\%$ of the selling pri find the selling price.

10. If the cost of an article is $\$a$ and the selling price is $ find the gain per cent.

11. A selling price of 25% above cost becomes 20% ab cost when the cost is raised by \$5. Find the cost.

12. A man saves \$10 a week after paying his living expens If he receives an increase of 12% in wages and the cost of livi increases by 15%, he would still save \$10 a week. Find his wee wage.

13. A 5% solution is made by dissolving 5 lb. of salt in 95 of water. How much water should be added to make this a 4 solution?

14. To 100 lb. of a 4% solution of salt it is desired to a enough salt to make it a 10% solution. How much salt should added?

15. A bag of grass seed is 60% timothy and the rest clov When 30 lb. of clover are added, the mixture becomes half a half. Find the original weight of the bag.

16. A sporting-goods merchant sells tennis racquets at advance of 25% above cost. If the cost of each racquet sho rise by \$2.50, his old selling price would be $11\frac{1}{9}\%$ above c Find the cost price.

17. A man lends $\$c$, part of it at $a\%$ and the balance at b If his income from each loan is the same, find the number dollars in each.

18. A merchant sold for \$232 some goods which he had mar at \$250. He sold some of the goods at a discount of 10% and rest at a discount of 5%. What fraction of the goods did he at each discount?

19. Of a sum of \$1000 how much should be invested at 5% d how much at 3% in order to make $4\frac{1}{2}$% on the whole in- stment?

20. Of a sum of \$100 how much should be invested at a% and w much at b% in order to make c% on the whole investment, ere c is a number between a and b?

Problems involving lengths, areas, and volumes.

ample. A rectangular garden 32 feet by 24 feet is designed h a central lawn surrounded by a perennial border of uniform lth. The area of the border is equal to that of the lawn. Find width of the border.

Let the width of the border be x feet. Then the dimensions of the lawn are $32 - 2x$ feet and $24 - 2x$ feet. Hence the area of the lawn is $(32 - 2x)(24 - 2x)$ square feet. Since this is half the whole area,

$$(32 - 2x)(24 - 2x) = \tfrac{1}{2} \times 32 \times 24$$

which gives $(16 - x)(12 - x) = \tfrac{1}{2} \times 32 \times 6 = 96.$

Hence $192 - 28x + x^2 = 96,$

or $x^2 - 28x + 96 = 0,$

or $(x - 4)(x - 24) = 0.$

Therefore $x = 4$ or $24.$

It follows that the width of the border is ***either*** 4 feet ***or*** feet. Let us examine these two answers. If the border is 4 feet le, the dimensions of the lawn are $32 - 8$ feet and $24 - 8$ feet, that the area of the lawn is 24×16 square feet. The total a of the garden is 32×24 or 768 square feet, which is twice area of the lawn. This verifies the answer 4 feet.

Now consider the second answer. It is clear that in a garden feet by 24 feet it is impossible to have a border 24 feet wide und a central lawn. Hence this answer is ***inadmissible.***

The fact that our equation led us to two answers of which only one is issible serves to emphasize the reasoning involved. This reasoning is as ws. ***If*** there is a border satisfying the conditions of the problem, ***then***

its width is ***either*** 4 feet ***or*** 24 feet. This does not say that the width mu be 4 feet or that it must be 24 feet, but merely that it cannot be any oth number. Moreover, it does not say that either 4 feet or 24 feet will satis the conditions of the problem. By substituting in the problem we find th 4 feet satisfies the conditions and 24 feet does not.

This example shows that the equation by which we solve a problem son times contains more information than the problem. Consider the followi problem:

How many quarts of water have I in a pail if, when I add an equal amou and then empty out 2 quarts, I have left 3 times the original amount?

If the original amount is x quarts, the conditions of the problem give t equation

$$2x - 2 = 3x$$

whence $x = -2$. The equation has a solution but it is inadmissible for given problem, which obviously has no solution.

EXERCISES

1. One side of a rectangular 3-acre field is 4 rods longer th the other. Find the lengths of the sides.

2. The height of a triangle is equal to its base and its area 32 square inches. Find the base.

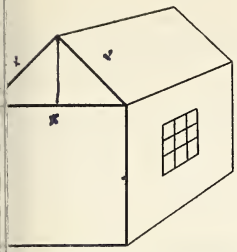
3. A rectangular building is to be designed to contain a flo space of 1200 square feet. If it is to be 10 feet longer than it wide, find the required dimensions.

4. Two rods are each 25 inches long. It is required to c each rod into two parts so that the four parts can be assembled form a rectangle with an area of 144 square inches. Find t lengths of the parts.

5. A square courtyard 40 feet to a side is surrounded by path of uniform width. If the area of the path is 7/9 that of t courtyard, find the width of the path.

6. A square mirror has a frame 3 inches wide. If the area the frame is 9/16 that of the mirror, find the side of the mirr

7. A formal pool twice as long as it is wide is surrounded b 4-foot border of lawn. The pool and lawn occupy an area 2304 square feet. Find the dimensions of the pool.



8. The end of a garage has a gable roof. The width of the garage is equal to the height of the eaves and the angle of the gable is a right angle. If the area of the end is 160 square feet, find the height of the peak above the ground. (*Hint*: The number which you choose to represent by x need not be the number ked for in the problem.)

9. Of two rectangular 10-acre fields A and B, one side of A is ods longer than one side of B and the other side of A is 4/5 of other side of B. Find the dimensions of the fields.

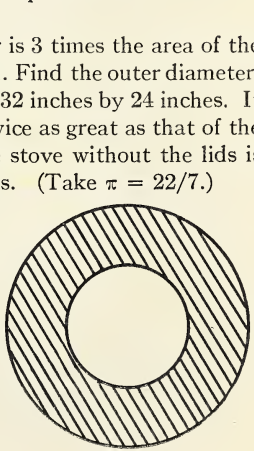
10. When a farmer has mowed eleven 6-foot swaths around his field, he finds that the field is half mowed. If his field is rectgular, with sides in the ratio 3:4, find the number of acres in field.

11. A printed page 8 inches by 5 inches has margins at the and bottom which are $1\frac{1}{2}$ times as wide as those at the sides. e area covered by the printing is 26 square inches. Find the lth of the margins.

12. The area of one face of a washer is 3 times the area of the e and the width of the ring is $\frac{1}{2}$ inch. Find the outer diameter.

13. The top of a stove is a rectangle 32 inches by 24 inches. It s four lids of which two have radii twice as great as that of the er two. The area of the top of the stove without the lids is $\frac{4}{7}$ sq. in. Find the radii of the lids. (Take $\pi = 22/7$.)

14. A cylindrical silo 30 feet high s concrete walls 1 foot thick. If volume of concrete in the silo 270π cu. ft., find the inner radius.



15. When the edges of a cube are each increased by 1 inch, th volume is increased by 61 cu. in. Find the edge of the origin cube.

Problems involving time, rate, and distance. If a car trave for t hours at the rate of r miles per hour, it covers a distance d miles, where

$$d = rt.$$

This formula can also be written

$$r = d/t \qquad \text{or} \qquad t = d/r.$$

Other units may be employed so long as they are used co sistently in the formula.

Example. Two cars made a journey of 360 miles. The fir travelled on the average 6 miles per hour faster than the secon and completed the journey in 2 hours less time. Find the avera speeds of the two cars.

Let the average speed of the slower car be v miles per hou Then that of the faster car is $v + 6$ miles per hour. The tir taken by the slower car to travel 360 miles is $360/v$ hours a the time of the faster car is $360/(v + 6)$ hours. Hence we ha

$$\frac{360}{v} - \frac{360}{v+6} = 2.$$

It follows that $$180\left(\frac{1}{v} - \frac{1}{v+6}\right) = 1.$$

Therefore $$180(v + 6 - v) = v(v + 6),$$

or $$1080 = v^2 + 6v,$$

or $$v^2 + 6v - 1080 = 0,$$

whence $$(v - 30)(v + 36) = 0.$$

Hence $$v = 30 \text{ or } -36.$$

Since, by the nature of the problem, v must be a positi number, the root -36 is inadmissible. It follows that t speed of the slower car is 30 miles per hour and that of t faster car 36 miles per hour.

Verification: The time taken by the slower car is 360/30 hours, or 12 hours; and the time taken by the faster car is 360/36 hours, or 10 hours.

EXERCISES

Answer the first twelve orally

1. If a point travels at $4\frac{1}{2}$ feet per minute, how far does it go 12 minutes?
2. If the speed is 6 miles per hour and the distance covered is miles, give the time taken.
3. If $2\frac{1}{2}$ hours are required to cover a distance of 30 miles, d the speed.
4. For a journey of 100 miles, what is the time required if the erage speed is (*a*) 30 m.p.h., (*b*) x m.p.h., (*c*) $x + 6$ m.p.h.?
5. What is the average speed for a 250-mile trip if the time ken is (*a*) 10 hours, (*b*) t hours, (*c*) $x - 4$ hours?
6. How long does it take a team of horses walking at $4\frac{1}{2}$ miles hour to go 18 miles?
7. How far does a point go in (*a*) h hours at n feet per hour, h hours at p feet per minute?
8. How much time is taken to go $3c$ yards at $4c$ feet per ond?
9. What is the effect on the time taken if the speed is doubled ile the distance is fixed?
10. What is the effect on the distance covered if the speed is ubled while the time taken is fixed?
11. What is the effect on the time taken if the distance and speed are both doubled?
12. What is the effect on the distance covered if the speed and time are both doubled?
13. On a journey of 60 miles a motorist averaged 9 miles per ur less than his usual speed, as a result of which he took 20 nutes longer than his usual time to make the trip. Find his ed.

$v_s = 36$ $v_f = 40$

14. Two cars leave at the same time and place for a tow 90 miles distant. One car travels 4 miles an hour faster than th other and arrives a quarter of an hour earlier. Find their speed

15. A's rate of travelling is one mile an hour less than B rate, and B's time in going 21 miles is 20 minutes less than A time in going 20 miles. Find A's rate of travelling.

16. A man who was removing from one town to anothe loaded his furniture on a transport which left at 9 a.m., travellin at 20 m.p.h. At 10 a.m. the owner left by car, travelling a 45 m.p.h. Where and when did he pass the transport?

17. A man walks from A to B at a certain rate. He returns b another road which is 6 miles longer and his rate of walking $\frac{1}{2}$ mile per hour less. The total distance walked was 48 mil and the total time of walking was 15 hours. Find the rate o walking in each direction.

18. Two towns on opposite sides of a lake are 28 miles apa by water. At 6 a.m. a boat leaves each town to go to the othe The boats pass each other at 8 a.m. One boat arrives at i destination 1 hour and 10 minutes earlier than the other. Fin the speed of each boat.

Miscellaneous problems. We make no attempt to classify a the problems which can be solved by means of equations. Th exercises which follow contain a considerable variety. By way o contrast with the previous types we add here one more examp which directs attention to the following facts regarding number

The number $85 = 8 \times 10 + 5$. Likewise the number who tens and units digits are a and b is $10a + b$.

Similarly the number $346 = 3 \times 100 + 4 \times 10 + 6$ and th 3-digit number whose digits from left to right are a, b, and c $100a + 10b + c$.

Example 1. Write a formula for a number whose tens digit twice its units digit and show that the number contains th factors 7 and 3.

If the units digit is a, then the tens digit is $2a$, and the number is

$$\begin{aligned} 2a(10) + a &= 20a + a, \\ &= 21a, \\ &= 7 \times 3 \times a. \end{aligned}$$

Hence the number contains the factors 7 and 3.

ample 2. In a number with two digits, the tens digit exceeds units digit by 2. The sum of this number and the number ained by interchanging the digits is 132. Find the number.

Let the units digit be x. Then the tens digit is $x + 2$ and the number is

$$10(x + 2) + x.$$

The number formed by interchanging the digits is

$$10x + (x + 2).$$

Since the sum of the numbers is 132, we have

$$22x + 22 = 132$$

from which $$x + 1 = 6$$

and $$x = 5.$$

It follows that the required number is 75. This answer is verified by the fact that $75 + 57 = 132$.

EXERCISES ON CHAPTER VII

A

ve the equations 1 to 25:

1. $\dfrac{5}{x} = \dfrac{2}{3}$.

2. $\dfrac{3}{x} = 6 + \dfrac{1}{x}$.

3. $\dfrac{3}{x} + 2 = \dfrac{15}{x} + 1$.

4. $\dfrac{x+1}{4} = \dfrac{2x-2}{7}$.

5. $\dfrac{4}{x+3} = \dfrac{3}{x+4}$.

6. $\dfrac{x}{x+1} = \dfrac{x+1}{x+3}$.

7. $ax - b = bx + a$.

8. $\dfrac{x}{a} + \dfrac{a-x}{b} = 1$.

9. $\frac{px+q}{r} = \frac{qx+p}{s}$.

10. $x^2 - 7x = 0$.

11. $x^3 - 9x = 0$.

12. $\frac{4}{x} = \frac{x}{9}$.

13. $8x - 2(3x - 5) = 5(4x - 7) - 3x$.

14. $3(2x + 3) - 4(3x + 4) = 5(2 - 3x) - 7(3 - x)$.

15. $\frac{3}{8}(3x + 2) = \frac{5}{2}(9 - x)$.

16. $0.6x - 2.8 = 2.4x - 3.4$.

17. $5(0.3x - 1) + 1.2x = 0.8(4x - 5)$.

18. $(7x - 3)(5x + 2) = 0$.

19. $(3y + 4)(2y - 5)(4y + 9) = 0$.

20. $t^2 - t - 30 = 0$.

21. $x^2 + 2x - 15 = 0$.

22. $z^2 + z - 132 = 0$.

23. $10s^2 - 11s + 1 = 0$.

24. $2x^2 - 5x - 12 = 0$.

25. $6r^2 - 5r - 6 = 0$.

26. Find three consecutive integers whose sum is 33.

27. Find two numbers which differ by 5 and are in the ra 3 : 4.

28. The denominator of a fraction exceeds the numera by 8. If each of the terms is increased by 7, the fraction redu to 3/4. Find the fraction.

29. What number exceeds its square by half of itself?

30. If trebling a number and squaring it give the same res what is the number?

31. One number exceeds another by 25. If 10% of the firs equal to 15% of the second, find the numbers.

32. One loan bears interest at 4% per annum and a sec loan, which is \$60 less, bears interest at $4\frac{1}{2}$%. A year's inte on the two loans together is \$12.60. Find the loans.

33. In an election contested by two candidates, the win received 56% of the total number of votes cast and his major was 456. Find the number of votes cast.

34. The length of a rectangular field exceeds the width 10 rods. If each side were increased by 10 rods the area would increased by 600 square rods. Find the dimensions of the field

35. The length of a rectangle is twice its width. When each e is increased by 1 unit, the area is increased by 10 square its. Find the original area.

36. Circular ripples are formed on a smooth pond by dropping tone into it. When the radius of a ripple has increased by oot, the enclosed area has increased by 9π square feet. Find first radius.

37. When the price of handkerchiefs was increased 5c. each, vas found that a number of handkerchiefs sold at the old price ught the same return as 7/8 of that number sold at the new ce. Find the original price.

38. A owes a sum of money on which he pays interest at 5%. wes \$50 more than A but he pays interest at 3%. If A and B the same interest, find how much they owe.

39. A train travelling at an average speed of 40 m.p.h. takes minutes longer to complete a run than a second train travelling an average speed of 50 m.p.h. Find the length of the run.

40. A man made a journey of 12 miles and returned, in 7 hours. speed on the return journey was $\frac{3}{4}$ of that on the outward rney. Find his speeds.

41. A 2-digit number is equal to 4 times the sum of its digits the units digit exceeds the tens digit by 3. Find the number.

42. The sum of the digits in a 2-digit number is 10 and the nber exceeds by 36 that formed by interchanging the digits. d the number.

43. How much salt should be dissolved in 100 pounds of water make a 10% solution by weight?

44. A wheel makes 200 revolutions per minute. How many olutions does it make in m minutes? How long does it take r revolutions? Express its rate in degrees per second.

B

ve the equations 45 to 62:

45. $\frac{1}{2}(x-2) - \frac{1}{3}(x-3) + \frac{1}{4}(x-4) = 4.$

46. $\dfrac{5x-4}{6} - \dfrac{x+2}{4} = \dfrac{4x-5}{9} + \dfrac{1}{2}.$

47. $\dfrac{3x+4}{25} - \dfrac{2x+6}{15} + \dfrac{x-4}{9} = 0.$

48. $\dfrac{3x-1}{5} - \dfrac{4x+2}{15} = \dfrac{5x+1}{6} - \dfrac{7x-9}{10}.$

49. $\frac{2}{3}(2x-5) - \frac{3}{4}(5x+3) = \dfrac{7x}{6} - 2.$

50. $\dfrac{3x+8}{12} - \frac{2}{3} = \dfrac{5x+2}{15} + \frac{1}{5}.$

51. $\dfrac{x+9}{2} + \dfrac{3x+1}{10} = \dfrac{4x-6}{5} + \dfrac{12x-3}{15}.$

52. $\dfrac{x+3}{x+2} + \dfrac{2x+1}{3x+1} = \frac{5}{3}.$

53. $\dfrac{4}{3x+3} = \frac{1}{6} + \dfrac{3}{2x+2}.$

54. $0.5(2x-5) - 0.2(x-2.3) = 0.25(3x-7.2).$

55. $0.8(6x-4.5) - 0.3(3x+1.5) = 1 + 0.125(5x-1.1).$

56. $(3x-5)^2 - 16 = 0.$

57. $(x+1)^2 - 7(x+1) + 12 = 0.$

58. $(2x+3)^2 + 3(2x+3) - 10 = 0.$

59. $6(5-y)^2 + (5-y) - 12 = 0.$

60. $(x^2-4)(x^2-9) = 0.$

61. $(x^2-a^2)(x^2-b^2)(x^2-c^2) = 0.$

62. $\dfrac{3}{x-5} + \dfrac{2x}{x-3} = 5.$

63. (*a*) What number is the negative of its square? (*b*) Wh number is twice its square?

64. What numbers are numerically (*a*) greater, (*b*) less th their squares?

65. The inner and outer radii of a hollow tile are r_1 and inches respectively and the volume of material in the tile V cu. in. Find the length of the tile.

6. A circular running track a feet wide has the same area as a le with radius $5a$ feet. Find the inside length of the track.

7. A rectangle is twice as long as it is wide. A second rectgle of the same area is a feet longer and b feet narrower than first. Find the dimensions of the original rectangle.

8. A truck left a town at 1 p.m., travelling at 25 m.p.h. At p.m. a car left on the same highway travelling at 40 m.p.h. ere and when did the car pass the truck?

9. A calendar h inches long is designed with a picture a in. b in. and a calendar pad c in. by d in. The picture and pad ether take up half the area of the calendar. Find the width of calendar.

10. A door h feet high is surrounded on the sides and top by a d moulding 5 inches wide. If the area of the moulding is half t of the door, find the width of the door.

11. An increase of \$10 in the cost of an article and a decrease 10 in the selling price changes a gain of 10% of cost into a of 10% of cost. Find the original cost.

12. In a 2-digit number the tens digit exceeds the units digit 4. Twice the number obtained by reversing the digits is 10 than the original number. Find the original number.

13. A rectangle has an area of 90 square inches and its unal sides differ by 9 inches. Find the sides.

14. The altitude of a triangle is 3 inches greater than the base the area is 27 square inches. Find the base.

15. If two opposite sides of a square are increased by 1 inch le the other two sides are decreased by 1 inch, will the area be reased or decreased and how much?

16. When two opposite sides of a square are increased by ches and the other two sides decreased by 4 inches, the lting rectangle has half the area of the square. Find the area he square.

17. In a 3-digit number the tens digit is 0 and the hundreds t exceeds the units digit by 4. The sum of the original number the number obtained by reversing the digits is 1010. Find number.

78. In a 3-digit number the units and hundreds digits a equal and the sum of the three digits is 17. The number obtain by interchanging the tens and hundreds digits exceeds th original number by 450. Find the number.

79. Is it possible to arrange 900 soldiers in a hollow squa with 50 men to a side of the outer square? If so, how many ran of soldiers will there be?

80. In a journey of 55 miles the number of miles per hour the average speed was 6 less than the number of hours. Find th average speed.

81. Two towns A and B are 159 miles apart. At 9 a.m. motorist started from A, travelling towards B at 40 m.p.h. 10 a.m. another motorist left B, travelling towards A at 45 m.p. When and where do the two cars pass each other?

82. I spent the same amounts of money for postage stamps two different denominations, of which one was priced $2\frac{1}{2}$ tim that of the other. I received 60 more stamps of one kind than the other. How many of each kind did I buy?

C

83. $x^2 - x - \cdots = 0$. Write this equation 3 times, i serting in the blank space 3 different numbers each of which w make the equation have integral roots. Find the roots of each the resulting equations.

84. $x^2 + \cdots x - 24 = 0$. Write this equation 3 times, i serting 3 different coefficients of x which will make the equati have integral roots. Find the roots of each equation.

85. Solve (*a*) $\dfrac{3x-5}{2x+7} = 1$, (*b*) $\dfrac{3x-5}{2x+7} = 0$.

86. Has the equation $\dfrac{x+1}{x+2} = \dfrac{x+5}{x+6}$ any roots? Explain.

87. The equation $x^2 - px + q = 0$ has the roots m and Find p and q in terms of m and n.

88. Find the four roots of $x^4 - 13x^2 + 36 = 0$.

89. Find the four roots of $(x^2 - x)^2 - 18(x^2 - x) + 72 = 0$.
90. Find the four roots of $(x^2+x+1)^2-16(x^2+x)+23=0$.
91. Solve $hx^2 - (h + k)x + k = 0$.
92. Solve $(a - b)x^2 + (b - c)x + c - a = 0$.
93. In a right triangle the sides about the right angle differ 2 inches and the hypotenuse is 10 inches. Find the sides.
94. Choose any two numbers and carry out the following ps. (*a*) Add the two numbers. (*b*) Multiply the sum by the t number. (*c*) Multiply the sum by the second number. Add the results of (*b*) and (*c*). (*e*) Square the result of (*a*). hat do you observe of the results of (*d*) and (*e*)? Is this an ident? Give an algebraic explanation.
95. Choose any two different numbers and carry out the owing steps. (*a*) Square the smaller number. (*b*) Add the o chosen numbers. (*c*) Subtract them. (*d*) Multiply the nbers resulting from (*b*) and (*c*). Add the result of (*a*) to that (*d*). (*f*) Take the square root of this sum. What relation has final number to your original numbers? Is this an accident? e an algebraic explanation.

CHAPTER VIII

SQUARE ROOTS

Operations and Inverse Operations

When, in arithmetic or algebra, we add, subtract, multipl or divide, we say that we perform an arithmetic or algebra ***operation.*** The word *operation* is a convenient term for great many 'things to do' in mathematics. Taking reciproc of numbers and taking powers of numbers are two other exampl of mathematical operations.

Now consider some of these operations a little more closel Subtraction is sometimes said to be the ***inverse*** of additio when you subtract 4 from 9, getting 5, you undo what was do when 4 and 5 were added to get 9. Also, division is the inver of multiplication; you see the inverse relation betwe $12 \div 3 = 4$ and $3 \times 4 = 12$.

Consider the operation of raising a number to a power. A example is $15^2 = 225$. Here we begin with 15 and find square. To perform the inverse operation we should begin wi 225 and find its ***square root,*** that is a number which has 2 for its square. The result is expressed by

$$\sqrt{225} = 15.$$

There is, as we know, a second number, -15, which gives 2 when squared. Hence 225 has a second square root,

$$-\sqrt{225} = -15.$$

A square root of a number, n, is a number which has n for its square.

The operation of taking roots can be further extended. ce $4^3 = 64$, we say that the ***cube root*** of 64 is 4, or

$$\sqrt[3]{64} = 4.$$

o $3^4 = 81$ and $2^5 = 32$, whence

$$\sqrt[4]{81} = 3 \text{ and } \sqrt[5]{32} = 2.$$

EXERCISES

For oral answers

Give the positive value of:

1. $\sqrt{10{,}000}$.	2. $\sqrt{1{,}000{,}000}$.	3. $\sqrt{169}$.
4. $\sqrt{8100}$.	5. $\sqrt{1{,}210{,}000}$.	6. $\sqrt{4/9}$.
7. $\sqrt{49/64}$.	8. $\sqrt{1.44}$.	9. $\sqrt{0.01}$.
10. $\sqrt{0.0001}$.	11. $\sqrt[3]{27}$.	12. $\sqrt[3]{8000}$.
13. $\sqrt[3]{0.064}$.	14. $\sqrt[4]{16/81}$.	15. $\sqrt[4]{160{,}000}$.
16. $\sqrt[4]{625}$.	17. $\sqrt[5]{100{,}000}$.	18. $\sqrt[5]{1/32}$.

Square Roots of Single Terms

The operation of finding square roots applies in algebra as l as in arithmetic. Since

$$(ab^2)^2 = a^2b^4,$$

ce the two square roots of a^2b^4 are

$$\sqrt{a^2b^4} = ab^2 \text{ and } -\sqrt{a^2b^4} = -ab^2.$$

shall for the present confine attention to one of the square s and omit the $-$ sign.

You have learned from the laws of indices that

$$(abc)^n = a^nb^nc^n \text{ and } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

When we use $n = \frac{1}{2}$, we get

$$(abc)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}} \quad \text{or} \quad \sqrt{abc} = \sqrt{a}\sqrt{b}\sqrt{c}$$

and $$\left(\frac{a}{b}\right)^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} \quad \text{or} \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

The square root of a product (quotient) of two expressions is product (quotient) of the square roots of the expressions.

Also, from the law of indices, we have

$$\sqrt{a^m} = (a^m)^{\frac{1}{2}} = a^{\frac{m}{2}}.$$

Example 1. $\sqrt{16a^4b^6c^{-2}} = 4a^2b^3c^{-1}$.

Example 2. $\sqrt{\dfrac{25a^{\frac{2}{5}}b^{\frac{4}{3}}}{49x^{\frac{6}{5}}y^{\frac{8}{3}}}} = \dfrac{5a^{\frac{1}{5}}b^{\frac{2}{3}}}{7x^{\frac{3}{5}}y^{\frac{4}{3}}}$.

EXERCISES

For oral answers

Give the square roots of:

1. $36a^4$.
2. $100a^2b^6$.
3. $81x^{-2}$.
4. $\dfrac{16}{y^8}$.
5. $4x$.
6. x^3.
7. $9xy^2$.
8. $25a^{-1}$.
9. $x^{\frac{2}{3}}$.
10. $\dfrac{x^{\frac{1}{3}}}{4}$.
11. $x^{\frac{2}{5}}y^{\frac{4}{5}}$.
12. $64x^{-4}y^{-5}$.
13. $0.01a^{-0.6}$.
14. $\dfrac{a^{-\frac{4}{3}}}{b^{-2}}$.
15. $p^{\frac{1}{2}}q^{\frac{1}{3}}$.
16. $\dfrac{0.04m^{\frac{2}{9}}n^{-\frac{4}{9}}}{p^{\frac{6}{7}}q^{-\frac{10}{7}}}$.
17. $\dfrac{a^{\frac{3}{5}}b^{-\frac{2}{5}}}{1.21}$.
18. $3^{2n}\,4^{-m}$.

Square Roots of Polynomials

We get information about the ***inverse*** process of finding
uare roots by analysing the ***direct*** process of squaring. The
owing formulas are useful:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2bc - 2ca$$

th these formulas as guides it is often possible to recognize an
ression which is the square of a binomial or a trinomial and
ce to find its square root by inspection. Consider the
ression.

$$9x^2 - 30xy + 25y^2.$$

ou recognize the square root and verify it, you may omit the
owing discussion. * * * The first term of this expression
he square of $3x$ and the third term the square of $5y$. The
ldle term is twice the product of $3x$ and $5y$. Since the sign
he middle term is negative, the square root is obtained by
ig a $-$ sign between $3x$ and $5y$. Hence

$$\sqrt{9x^2 - 30xy + 25y^2} = 3x - 5y.$$

could equally well have taken $5y - 3x$, which is the second
are root.

As a next example, consider

$$4x^2 + 9y^2 + 16 + 12xy - 16x - 24y.$$

Try to write the square root of this expression and verify it.

* * *

The first 3 terms are the squares of $2x$, $3y$, and 4 respectively. Are
other 3 terms twice the products taken in pairs? It is easy to see that
are. Finally, what signs should be used? Since the sign of $12xy$ is
the signs $2x$ and $3y$ must be alike, and since the sign of the other two
luct terms is $-$, the sign of 4 must differ from those of $2x$ and $3y$. It
ws that

$$\sqrt{4x^2 + 9y^2 + 16 + 12xy - 16x - 24y} = 2x + 3y - 4.$$

It should be noted that, when an expression is found to be a perfect square, this fact furnishes the means of factoring it since it is the product of two factors each equal to its square root.

EXERCISES

By inspection, write the square roots of the following. Hence factor each of the first ten expressions.

1. $16a^2 + 56a + 49$.
2. $36b^2 - 60b + 25$.
3. $64x^2 - 48xy + 9y^2$.
4. $\frac{4}{9}x^4 - \frac{2}{3}x^2y + \frac{1}{4}y^2$.
5. $\frac{p^2}{9} - \frac{pq}{3} + \frac{q^2}{4}$.
6. $\frac{1}{4}a^2b^2 + ab + 1$.
7. $100m^2 + 25n^2 + 100mn$.
8. $x^{-2} - 18x^{-1} + 81$.
9. $9x^2 + 16y^2 + 25z^2 - 24xy - 40yz + 30xz$.
10. $100m^4 + 16n^4 + 36 - 80m^2n^2 + 120m^2 - 48n^2$.
11. $x^4 + x^{-4} + 2$.
12. $\frac{4}{25}c^2 + \frac{9}{16}d^2 + \frac{25}{4} + \frac{3}{5}cd - 2c - \frac{15}{4}d$.
13. $a^{\frac{2}{3}} + 12a^{\frac{1}{3}}b^{\frac{1}{3}} + 36b^{\frac{2}{3}}$.
14. $28a^{\frac{1}{2}}b^{\frac{1}{2}} - 12b^{\frac{1}{2}}c^{\frac{1}{2}} - 42a^{\frac{1}{2}}c^{\frac{1}{2}} + 49a + 4b + 9c$.
15. $a^2 + b^2 - a - b + 2ab + \frac{1}{4}$.
16. $9a^4b^2 + 25b^4c^2 + 49c^4a^2 - 30a^2b^3c + 70ab^2c^3 - 42a^3$
17. $x^{\frac{2}{5}}y^{\frac{4}{5}} - 2\sqrt{5}\,x^{\frac{1}{5}}y^{\frac{2}{5}} + 5$.
18. $81x^{\frac{2}{3}} + 64y^{\frac{1}{2}} + 36z^{\frac{2}{5}} - 144x^{\frac{1}{3}}y^{\frac{1}{4}} - 96y^{\frac{1}{4}}z^{\frac{1}{5}} + 108x^{\frac{1}{3}}z$
19. $9(x^2 + 1)^2 - 24(x^2 + 1)y + 16y^2$.
20. $4a - b^{-1} - 2ab^{-1} + 4a^2 + 1 + 4^{-1}b^{-2}$.
21. $a^2b^{-2} + 4b^2c^{-2} + 25c^2a^{-2} - 4ac^{-1} - 20a^{-1}b + 10b$
22. $x^8 + 2x^7 + x^6 + 2x^5 + 2x^4 + x^2$.

The Square Root of a Polynomial
A General Method

For a polynomial such as

$$x^4 + 4x^3 + 10x^2 + 12x + 9$$

is not obvious, on inspection, whether or not it is the square of other polynomial and, if so, what is the square root. In order obtain a formal process for finding such square roots let us alyse the operation of squaring a binomial. We have

$$\begin{aligned}(x + y)^2 &= x^2 + 2xy + y^2 \\ &= x^2 + (2x + y)y.\end{aligned}$$

Now begin with the polynominal $x^2 + 2xy + y^2$. We wish construct a process for finding its square root, which we eady know to be $x + y$. The first term in the root is the uare root of x^2, the first term in the polynomial. Hence in lding up the square root we write the first term x, then square and subtract x^2 from the polynomial. The remainder is

$$2xy + y^2 \text{ or } (2x + y)y.$$

know from our formula that the second term of the square t is y. This is obtained by dividing $2x + y$ into the remainder $+ y^2$. The key to the process therefore consists in building the divisor $2x + y$. To do this we double the first term in square root, getting $2x$; we then divide $2x$ into $2xy$, getting This y is added to x to give the second term in the square t; it is also added to $2x$ to complete the divisor. The complete isor is now multiplied by the second term, y, of the root, ng the remaining terms of the polynomial. We may set down steps in a scheme somewhat resembling long division.

$$\begin{array}{r|l} x + y & \\ \hline x^2 + 2xy + y^2 & \\ x^2 & \\ \hline + y \mid 2xy + y^2 & \\ 2xy + y^2 & \\ \hline 0 & \end{array}$$

(1) Find the square root of x^2. Square this root and subtract from the polynomial.

(2) Double the first term, x, of the root for a ***trial divisor.*** Divide the trial divisor into $2xy$, getting the second term, y, of the root.

(3) Add y to the root and to the trial divisor. Multiply the complete divisor by y.

Consider now a polynomial with 5 terms.

$$4x^4 - 12x^3 + 29x^2 - 30x + 25.$$

The method used above for finding the square root of a trinomi admits a direct extension to the square root of this polynomi Examine the work as it is set down here and see if you c justify each step.

$$\begin{array}{r|l} & 2x^2 - 3x + 5 \\ \hline & 4x^4 - 12x^3 + 29x^2 - 30x + 25 \\ & 4x^4 \\ \hline 4x^2 - 3x & - 12x^3 + 29x^2 \\ & - 12x^3 + 9x^2 \\ \hline 4x^2 - 6x + 5 & 20x^2 - 30x + 25 \\ & 20x^2 - 30x + 25 \\ \hline & 0 \end{array}$$

You observe that, when we have obtained the two ter $2x^2 - 3x$ in the square root, we treat these as a single term a repeat the process already used. This means that we multi $2x^2 - 3x$ by 2 to obtain the next trial divisor, $4x^2 - 6x$. Si the $4x^2$ divided into $20x^2$ gives the quotient 5, we add 5 to the complete divisor $4x^2 - 6x + 5$, and also add the term 5 the square root. When the complete divisor is now multipl by 5 the remainder is 0, which shows that the given expres is the square of $2x^2 - 3x + 5$.

The explanation of the whole process can now be conden into the following form:

$$\begin{aligned} (2x^2 - 3x + 5)^2 &= (2x^2 - 3x)^2 + 5(4x^2 - 6x + 5) \\ &= 4x^4 - 3x(4x^2 - 3x) + 5(4x^2 - 6x + 5) \end{aligned}$$

The three steps of the process correspond to the three term this expression.

It is now clear that the method will apply to find the squ roots of polynomials of still higher degrees. If the process le to a final remainder of 0, the original polynomial is calle

rfect square. An expression written at random is very unlikely to a perfect square. The square root process will determine ether or not it is. The following example illustrates the ocess when negative and fractional indices are involved.

ample. Find the square root of

$$x^2 + 10x + 25 + 16x^{-1} - 4x^{\frac{3}{2}} - 20x^{\frac{1}{2}} - 24x^{-\frac{1}{2}}.$$

A first step here is to arrange the expression in descending vers of x.

$$\begin{array}{l}
\qquad\qquad x - 2x^{\frac{1}{2}} + 3 - 4x^{-\frac{1}{2}} \\
\hline
x^2 - 4x^{\frac{3}{2}} + 10x - 20x^{\frac{1}{2}} + 25 - 24x^{-\frac{1}{2}} + 16x^{-1} \\
\underline{x^2} \\
\hline
2x - 2x^{\frac{1}{2}}) \quad - 4x^{\frac{3}{2}} + 10x \\
\qquad\quad \underline{- 4x^{\frac{3}{2}} + \ 4x} \\
\hline
2x - 4x^{\frac{1}{2}} + 3 \ ) \quad 6x - 20x^{\frac{1}{2}} + 25 \\
\qquad\qquad\qquad\quad \underline{6x - 12x^{\frac{1}{2}} + \ 9} \\
\hline
2x - 4x^{\frac{1}{2}} + 6 - 4x^{-\frac{1}{2}}) - \ 8x^{\frac{1}{2}} + 16 - 24x^{-\frac{1}{2}} + 16x^{-1} \\
\qquad\qquad\qquad\quad - \ 8x^{\frac{1}{2}} + 16 - 24x^{-\frac{1}{2}} + 16x^{-1} \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

EXERCISES

d the square roots of the following expressions.

1. $9x^4 - 30x^3 + 37x^2 - 20x + 4.$
2. $64x^4 + 48x^3 - 87x^2 - 36x + 36.$
3. $49c^6 - 56c^5 + 58c^4 - 24c^3 + 9c^2.$
4. $25a^4 - 70a^3b + 89a^2b^2 - 56ab^3 + 16b^4.$
5. $36a^4b^4 - 60a^3b^3 - 23a^2b^2 + 40ab + 16.$
6. $81m^8 + 180m^6 - 98m^4 - 220m^2 + 121.$
7. $4a^6 - 4a^5 + 13a^4 - 22a^3 + 17a^2 - 24a + 16.$

8. $x^6 + 4x^5 + 4x^4 - 10x^3 - 20x^2 + 25$.

9. $4y^6 - 12y^4 + 28y^3 + 9y^2 - 42y + 49$.

10. $4p^6 - 20p^5 + 25p^4 + 24p^3 - 60p^2 + 36$.

11. $9k^6 - 24k^5 + 46k^4 - 52k^3 + 41k^2 - 20k + 4$.

12. $100p^2 + 180p + 281 + 180p^{-1} + 100p^{-2}$.

13. $a^3 + 4a^2 + 2a - 8 - 7a^{-1} + 4a^{-2} + 4a^{-3}$.

14. $x^{\frac{4}{3}} - 8x + 26x^{\frac{2}{3}} - 40x^{\frac{1}{3}} + 25$.

15. $a^{-\frac{4}{5}} + 12a^{-\frac{3}{5}} + 28a^{-\frac{2}{5}} - 48a^{-\frac{1}{5}} + 16$.

16. $16a^{-6} - 48a^{-4} + 108a^{-2} - 108 + 81a^2$.

17. $121n^{-2} - 176n^{-3} + 218n^{-4} - 310n^{-5} + 193n^{-6} - 126n^{-7} + 81n^{-8}$.

18. $25x - 60x^{\frac{1}{2}} + 106 - 84x^{-\frac{1}{2}} + 49x^{-1}$.

19. $16x^{\frac{2}{3}} - 56x^{\frac{1}{2}} + 9x^{\frac{1}{3}} + 70x^{\frac{1}{6}} + 25$.

20. $9y^{-\frac{6}{5}} - 24y^{-1} + 46y^{-\frac{4}{5}} - 40y^{-\frac{3}{5}} + 25y^{-\frac{2}{5}}$.

21. $x^6 + 6x^4 + 4x^3 + 9x^2 + 12x + 4$.

22. $4y^3 - 12y^2 + 33y - 52 + 60y^{-1} - 48y^{-2} + 16y^{-3}$.

23. $4x^{-6} - 20x^{-5} + 37x^{-4} - 46x^{-3} + 49x^{-2} - 24x^{-1} +$

24. $x^8 + 4x^7 - 2x^6 - 4x^5 + 15x^4 - 44x^3 + 46x^2 - 40x +$

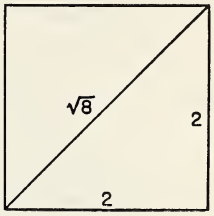
25. $9x - 30x^{\frac{3}{4}} + 13x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 80 - 44x^{-\frac{1}{4}} - 8x^{-\frac{1}{2}} - 48x^{-\frac{3}{4}} + 36x^{-1}$.

The Square Roots of Integers

The principle already used for finding square roots of po nomials can be extended to find square roots of integers. Obse first the following table of integers and their squares:

$1^2 = 1$ $10^2 = 1|00$ $100^2 = 1|00|00$ $1000^2 = 1|00|00|0$
$9^2 = 81$ $99^2 = 98|01$ $999^2 = 99|80|01$ $9999^2 = 99|98|00$

Taking $\sqrt{8}$ as an example, let us examine it more closely. a square has a side of 2 feet its diagonal $\sqrt{8}$ feet. This is a perfectly definite antity; *there is no inexactness about the mber* $\sqrt{8}$. It is only when we express e number in our decimal system of meration that we meet a difficulty. ie decimal system is not flexible ough to express this number in a minating form. For some purposes it is sufficient to work h the symbol $\sqrt{8}$. For some practical purposes, however, s is not sufficient. Suppose that a carpenter wished to fit a ice along the diagonal of a square opening which had a side of eet. Since he does not find $\sqrt{8}$ marked on his rule, he must ve it expressed in convenient fractions, such as decimal ctions.



It is possible to approximate to $\sqrt{8}$ as closely as we desire by lecimal. This is merely a matter of carrying out the process eady given for finding square roots. For finding the root rect to 2 decimal places the work is as follows:

$$
\begin{array}{r}
2.828 \\
\hline
8.0000 \\
4 \\
\hline
4\ 8\ )4\ 00 \\
3\ 84 \\
\hline
56\ 2\ )\ 16\ 00 \\
11\ 24 \\
\hline
564\ 8\ )\ 4\ 76\ 00
\end{array}
$$

$\sqrt{8} = 2.83$ (slightly too large)

Check: $2.83^2 = 8.0089$

ce we want the root correct to 2 decimal places, we carry out work until we see the magnitude of the figure in the third imal place, then round off the result to 2 places.

EXERCISES

Find, to 3 figures,

1. $\sqrt{263}$. 2. $\sqrt{6.42}$. 3. $\sqrt{8.37}$.

Find, to 4 figures,

4. $\sqrt{2}$. 5. $\sqrt{3}$. 6. $\sqrt{5}$.

Check your answers for $\sqrt{2}$ and $\sqrt{3}$ with the following, whi you will find it useful to memorize.

$$\sqrt{2} = 1.414$$
$$\sqrt{3} = 1.732$$

Find, to 4 figures,

7. $\sqrt{747}$. 8. $\sqrt{7.47}$. 9. $\sqrt{74.7}$.

(Since $7.47 = 747 \div 100$, it follows that $\sqrt{7.47} = \sqrt{747} \div$ Hence, when $\sqrt{747}$ has been found, $\sqrt{7.47}$ is obtained from by moving the decimal point one place to the left. On the otl hand, $\sqrt{74.7}$ must be found independently.)

10. $\sqrt{58.5}$. 11. $\sqrt{0.585}$. 12. $\sqrt{5.85}$.

13. $\sqrt{3.28}$. 14. $\sqrt{328}$. 15. $\sqrt{32.8}$.

16. $\sqrt{70.84}$. 17. $\sqrt{5.426}$. 18. $\sqrt{236.9}$.

EXERCISES ON CHAPTER VIII

A

1. How many digits are there in the integral part of square root of an integer which has (*a*) 8 digits, (*b*) 11 digits, 15 digits?

2. The first two digits of a number are 75. What is first digit in the square root if the number has (*a*) 5 digits be the decimal point, (*b*) 6 digits before the decimal point?

3. By inspection, write expressions whose squares are

(*a*) $49x^2 - 42ax + 9a^2$, (*b*) $81x^2 - 72xy + 16y^2$,

(*c*) $9c^2 + 25c^{-2} + 30$, (*d*) $4p^{-4} + 16p^{-6} + 16p^{-8}$.

4. By inspection, write the square roots of

(*a*) $a^2 + 4b^4 + 9c^6 - 4ab^2 + 6ac^3 - 12b^2c^3$,

(*b*) $9x^2 + 25y^2 + 49z^2 + 30xy + 70yz + 42xz$,

(*c*) $4p^2 + 25q^2 + 100r^2 + 20pq - 40pr - 100qr$.

nd the square roots of the expressions in Exs. 5 to 10.

5. $x^6 - 8x^5 + 20x^4 - 16x^3 + 4x^2$.

6. $9y^4 - 12y^3 - 38y^2 + 28y + 49$.

7. $81p^8 + 126p^6 - 41p^4 - 70p^2 + 25$.

8. $25x^6 - 30x^4 + 40x^3 + 9x^2 - 24x + 16$.

9. $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1$.

10. $49a^{10} - 70a^8 + 56a^7 + 25a^6 - 40a^5 + 16a^4$.

11. Evaluate $\sqrt{788{,}544}$.

12. Find, to 3 figures, the value of $\sqrt{768}$.

13. Find the side of a square whose area is 2809 sq. cm.

14. Find, correct to 3 figures, the side of a square whose area 347 sq. in.

15. A rectangle is designed with one side twice as long as other and with an area of 756 sq. in. Find its sides, each to nearest tenth of an inch.

16. Find, to the nearest tenth of an inch, the side of a square ich is equal in area to the sum of a rectangle 12 in. by 16 in. l a square of side 15 in.

B

17. Is it possible for an integer to be a perfect square if it ls in 2, 3, 7, or 8? Explain.

18. Given that $\sqrt{64516} = 254$, give without calculation the ues of as many as possible of the following: $\sqrt{6451.6}$, $\sqrt{645.16}$, $\sqrt{4.516}$, $\sqrt{6.4516}$, $\sqrt{0.64516}$, $\sqrt{0.064516}$.

19. Evaluate $\sqrt{47,169,424}$.

20. Find the first 4 digits in the decimal representation $\sqrt{54.63}$.

21. Find the first four digits in the decimal representation $\sqrt{546.3}$.

Find the square roots of the expressions, in Exs. 22 to 26.

22. $4z^6 - 12z^5 + 13z^4 - 26z^3 + 31z^2 - 10z + 25$.

23. $25a^{\frac{4}{3}} - 20ab^{\frac{1}{3}} - 56a^{\frac{2}{3}}b^{\frac{2}{3}} + 24a^{\frac{1}{3}}b + 36b^{\frac{4}{3}}$.

24. $49x^{-6}y^{-2} - 56x^{-5}y^{-3} + 58x^{-4}y^{-4} - 24x^{-3}y^{-5} + 9x^{-2}y$

25. $100h - 120h^{\frac{3}{4}}k^{\frac{1}{4}} + 136h^{\frac{1}{2}}k^{\frac{1}{2}} - 60h^{\frac{1}{4}}k^{\frac{3}{4}} + 25k$.

26. $4x^{-\frac{2}{3}} + 20x^{-1} + 53x^{-\frac{4}{3}} + 70x^{-\frac{5}{3}} + 49x^{-2}$.

27. Find, correct to a tenth of an inch, the diagonal of square whose side is 14.0 inches.

28. Find, correct to 1/100 of an inch, the diagonal of rectangle whose sides are 6.00 in. and 9.00 in.

29. Find, correct to 3 figures, the radius of a circle wh area is 154 sq. in.

30. A 3-acre field, in the shape of an isosceles triangle, is be fenced off in the corner of a rectangular pasture. Find, the nearest rod, the length of the two equal sides of the field.

31. Find the radius of a circle which is equal in area to sum of two circles with radii 3.2 in. and 4.1 in., respectively.

CHAPTER IX

RATIONAL AND IRRATIONAL NUMBERS

Rational Numbers

From the beginning of your study of numbers, you have met umbers of different kinds in approximately the following order:

(*a*) the positive integers, 1, 2, 3, · · · ;
(*b*) the positive fractions, such as 1/2, 2/3, 7/8, 13/4, · · · ;
(*c*) the integer 0;
(*d*) the negative integers and fractions, such as -1, -5, 2/5, $-22/7$, · · · .

Notice that every number of (*a*), (*b*), and (*d*) can be described as positive negative. The number 0, however, is neither positive nor negative.

All these numbers have the property that they can be ob-ined by beginning with the positive integers and applying to em only the four operations of ***addition, subtraction, ultiplication,*** and ***division.*** These will be called for brevity e ***four rational operations.*** For example, the negative eger -5 can be obtained by subtracting 8 from 3; 0 can be tained by subtracting 2 from 2; 3/4 can be obtained by dividing by 4, etc.

Any number which can be obtained in this manner can be pressed as a fraction of which the numerator and denominator integers, positive or negative. In order to convince yourself this fact, notice that

$$3 = \tfrac{3}{1}, \qquad 0 = \tfrac{0}{1}, \qquad -\tfrac{7}{3} = \tfrac{-7}{3}, \quad \text{etc.}$$

ain, take a complex fraction like

$$\frac{\frac{2}{5} - \dfrac{3 - \frac{1}{4}}{2\frac{1}{2}}}{4 - \frac{2}{3}},$$

which is built up from positive integers by the four ration operations. This fraction reduces to $\frac{-21}{100}$ or -0.21.

DEFINITION:

> ***Any number which has or can be reduced to the form P/Q, where P and Q are integers and Q ≠ 0, is called a rational number.***

The term ***rational*** is derived from the same Latin root as the word ***rati*** which is another name for fraction.

The important fact to remember is that, by beginning wi the positive integers and adding, subtracting, multiplying, a dividing any number of times and in any order, we shall alwa arrive at a rational number. You are reminded once more of t one exceptional case among these operations.

> *The one operation which we must avoid is that of dividing by* 0.

Any rational number can be expressed in decimal form. F example, $3/8 = 0.375$, $21/20 = 1.05$, $1/3 = 0.333 \cdots$ or 0 $4/11 = 0.363636 \cdots$ or $0.\dot{3}\dot{6}$.

> *When a rational number is expressed in decimal form, the result is either a terminating or a repeating decimal.*

EXERCISES

Express the following rational numbers in the form $P/$ where P and Q are integers:

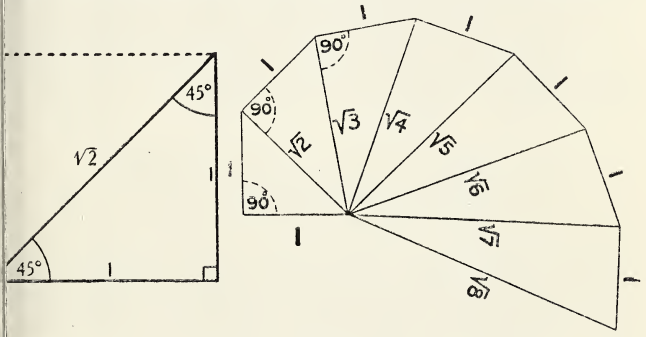
1. 24. 2. $8\frac{2}{5}$. 3. -6.1.
4. 16.25. 5. $-5.\dot{6}$. 6. $2.1\dot{4}$.

7. $\dfrac{3+\frac{1}{2}}{2-\frac{1}{3}}$. 8. $\dfrac{\frac{3}{5}-\frac{2}{3}}{\frac{5}{2}-\frac{7}{4}}$. 9. $\frac{3}{4}+\dfrac{\frac{7}{2}}{2-\frac{5}{8}}$.

10. $\frac{2}{7}(\frac{3}{4}-\frac{5}{9})-\frac{9}{16}(\frac{5}{6}-\frac{7}{10})$.

Have you ever used any numbers other than rational
umbers * * *
ou think at once of numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{2}$, etc.

Such numbers presented a challenge to mathematicians as long ago as 00 years. At that time arithmetic and algebra were in a very primitive state, rtly because no convenient notation had yet been invented. Greater progress d been made in geometry since geometrical figures are easily visualized and not require an elaborate notation for their study. The Greek mathema- ians of the school of Pythagoras (about 520 B.C.) were intensely interested the numbers which give the ratios of certain geometrical lines. If the side a square is 1 unit, then its diagonal is $\sqrt{2}$ units. Moreover, by means of ht-angled triangle constructions as illustrated in the figure, it is possible to nstruct lines whose lengths are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, . . . units, respectively.



It is a matter of interest that Pythagoras or his associates gave a proof at $\sqrt{2}$ is not a rational number. Their proof, although about 2500 years old, still a model of mathematical reasoning. In modern phraseology the proof is follows; you will recognize from your study of geometry that it is an example an indirect proof.

For the number $\sqrt{2}$, there are only two possibilities:

(1) $\sqrt{2}$ is a rational number;
(2) $\sqrt{2}$ is not a rational number.

Let us suppose that $\sqrt{2}$ is rational. Then it is a fraction p/q. Let us suppo also that this fraction has been reduced to its lowest terms by dividing out an common factors from the numerator and denominator. Then we have

$$\sqrt{2} = p/q,$$

where p and q have no common factor except 1.

By writing this equation in the form

$$\sqrt{2}\, q = p$$

and squaring both sides, we get

$$2\, q^2 = p^2.$$

Now 2 times q^2 is an integer which contains the factor 2; it is therefore an ev integer. Hence p^2 is an even integer and therefore p is even (since the squa of an odd integer is also odd). Since p contains the factor 2, then p^2 contai the factor 4. Hence $2q^2$ contains the factor 4 and q^2 contains the factor Again it follows that q must be an even number. Then, since p and q are bo even numbers, the fraction p/q can be further reduced by dividing out 2 fro p and q. But it is absurd that p and q can have no factors except 1 and th they are both divisible by 2. Therefore our first possibility (that $\sqrt{2}$ is rational number) is false. Hence the second possibility (that $\sqrt{2}$ is n rational) is true.

A number that is not rational is called ***irrational.***

If an integer is a perfect square, then its square root is a integer and therefore rational. Thus $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$, $\cdots$ are rational. The square roots of non-square integers are i rational. Similarly $\sqrt[3]{8}$, which equals 2, is a rational number b $\sqrt[3]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{4}$, and, in fact, the cube root of any number which not a perfect cube, are irrational. A similar statement applies fourth roots, fifth roots, and roots of higher orders.

Are there any irrational numbers which can not be expres in terms of roots of integers? * * * The fact is that there a such numbers and at least one of them has a long and interesti history. It is the number π which is the ratio of the circumferen

the diameter of a circle. As approximations for π, we use $3\frac{1}{7}$ 3.1416 or 3.14159, all of which are rational numbers. However, ne of these represents π exactly, nor does any root or rational mbination of roots.

An irrational number can be represented approximately by a decimal. ice any decimal which terminates or one which repeats is a rational number, follows that the decimal representation of an irrational number neither minates nor repeats. We find that

$$\sqrt{2} = 1.41421 \ldots$$

l that

$$\pi = 3.1415926 \ldots$$

hough $\sqrt{2}$ and π are both perfectly exact numbers, no one knows what 10000th digit in either number is. To find these digits would involve rodigious amount of work but, fortunately, it is difficult to imagine any vantage which the knowledge of these digits would give.

EXERCISES

(For oral answers)

ck out the irrational numbers in the following list:

1. 1.46.	**2.** $3.\dot{7}$.	**3.** $\sqrt{8}$.	**4.** $\sqrt{16}$.
5. $\sqrt[3]{27}$.	**6.** $\sqrt[3]{54}$.	**7.** $\sqrt[5]{32}$.	**8.** $5^{3/2}$.
9. $8^{2/3}$.	**10.** $1 + \sqrt{2}$.	**11.** $0.\dot{4}5\dot{6}$.	**12.** 0.
13. 17/18.	**14.** $1/\sqrt{6}$.	**15.** $(\sqrt[3]{4})^6$.	**16.** π^2.

Irrational Square Roots

The signs $\sqrt{}$, $\sqrt[3]{}$, or in general $\sqrt[n]{}$ are called root signs or ***dical signs.*** In origin, the sign $\sqrt{}$ is a modification of *r*, the tial letter of the Latin *radix*, meaning root. A root of a number, pressed with a radical sign, will be called a ***radical.***

If a is a rational number and $\sqrt{a}$ is irrational, the latter is sometimes led a ***quadratic surd.***

We confine our attention now to square roots. If these are tional, then the radical sign is unnecessary. Thus $\sqrt{36} = 6$,

$\sqrt{400} = 20$, etc. Moreover, any number can be written as radical; $17 = \sqrt{289}$, $1000 = \sqrt{1{,}000{,}000}$ etc.

Changing the form of a radical. Since $\sqrt{2}$ is a positi number whose square is 2, therefore

$$\sqrt{2} \times \sqrt{2} = 2 = \sqrt{4}.$$

Again, $\sqrt{4} \times \sqrt{25} = 2 \times 5 = 10 = \sqrt{100}$,
and similarly $\sqrt{9} \times \sqrt{16} = \sqrt{144}$.

Can $\sqrt{3} \times \sqrt{7}$ be found similarly by multiplying the numbe under the root signs? To answer this question let us square t product:

$$(\sqrt{3}\times\sqrt{7})^2 = \sqrt{3}\times\sqrt{7}\times\sqrt{3}\times\sqrt{7} = \sqrt{3}\times\sqrt{3}\times\sqrt{7}\times\sqrt{7} = 3\times 7 = 21.$$

Hence $\sqrt{3}\times\sqrt{7} = \sqrt{21}$.

It is easy to see that this rule works for all numbers. For, le and b be any two numbers and consider $\sqrt{a} \times \sqrt{b}$. We ha

$$\begin{aligned}(\sqrt{a} \times \sqrt{b})^2 &= \sqrt{a} \times \sqrt{b} \times \sqrt{a} \times \sqrt{b} \\ &= \sqrt{a} \times \sqrt{a} \times \sqrt{b} \times \sqrt{b} \\ &= ab,\end{aligned}$$

whence $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

The product of the square roots of two numbers is equal to the square root of the product of the numbers.

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

The following examples illustrate the application of this ru

Example 1. $\sqrt{4} \times \sqrt{6} = \sqrt{24}$.

The product $\sqrt{4} \times \sqrt{6}$ is conveniently written as $2 \times$ or $2\sqrt{6}$.

Hence $2\sqrt{6} = \sqrt{24}$.

ample 2. $3\sqrt{5} = \sqrt{9} \times \sqrt{5} = \sqrt{45}.$

ample 3. $9\sqrt{2} = \sqrt{81} \times \sqrt{2} = \sqrt{162}.$

For some purposes $3\sqrt{5}$ and $9\sqrt{2}$ are more convenient forms an $\sqrt{45}$ and $\sqrt{162}$. Hence we can simplify the form of a dical when the number under the root sign contains a perfect uare factor. The following three examples illustrate such nplification.

ample 4. $\sqrt{27} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}.$

ample 5. $\sqrt{72} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}.$

ample 6. $\sqrt{a^2b} = a\sqrt{b}.$

A square factor under a radical sign may be removed, provided the square root of the factor is placed before the radical.

A number like $\sqrt{27}$, with no factor before the radical sign, is led a ***pure*** or ***entire radical;*** a form like $3\sqrt{3}$, which is the oduct of a rational number and a pure radical, is called a ***ixed radical.***

The use of radicals in geometry. The following examples ow how radicals arise in computational work in geometry.

ample 1. Find the side of a square whose area is 54 sq. in.

If the side is s in., we have

$$s^2 = 54,$$

whence, $$s = \sqrt{54} = 3\sqrt{6},$$

By referring to the table of square roots, we find that

$$3\sqrt{6} = 3 \times 2.449 = 7.35, \text{ correct to 3 figures.}$$

The required side is $3\sqrt{6}$ in. or 7.35 in. approximately.

Example 2. Find the radius of a circle whose area is 88 sq. in.

If the radius is r in., we have

$$\tfrac{22}{7}r^2 = 88$$

or $$r^2 = \tfrac{7}{22} \times 88 = 28.$$

Hence, $$r = \sqrt{28} = 2\sqrt{7}.$$

The radius is $2\sqrt{7}$ in. or 5.29 in., to 3 figures.

Example 3. Find the diagonal of a field whose sides are 40 ro and 60 rods.

If the diagonal is d rods, then

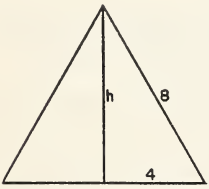
$$d^2 = 40^2 + 60^2 = 1600 + 3600 = 5200,$$

and $$d = \sqrt{5200} = \sqrt{400 \times 13} = 20\sqrt{13}.$$

From the tables we find that $20\sqrt{13} = 20 \times 3.6056 = 72.$ to 3 figures.

The required diagonal is $20\sqrt{13}$ rods or 72.1 rods appro mately. Notice that in giving the latter answer we are assumi that the original measurements were correct to within a ten of a rod.

Example 4. Find the area of an equilateral triangle whose si is 8 in.



If the altitude is h in., then the Pythagorean theorem

$$h^2 + 4^2 = 8^2$$

and

$$h^2 = 64 - 16 = 48.$$

Hence

$$h = \sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{}$$

The area is found from the formula $A = \frac{1}{2}bh$. Therefor

$$A = \tfrac{1}{2} \times 8 \times 4\sqrt{3} = 16\sqrt{3}.$$

Assuming that the data have 3-figure accuracy, this ans gives for the area 27.7 sq. in.

ample 5. By using the '*s*' formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

d the area, A, of the triangle whose sides are 6, 10, and 12 hes, respectively.

In this formula, s is the semi-perimeter of the triangle. The nputation is as follows:

$$\begin{array}{rr} a = 6 & s = 14 \\ b = 10 & s-a = 8 \\ c = 12 & s-b = 4 \\ \hline 2s = 28 & s-c = 2 \end{array}$$

Add: $2s = 28$ (Check)

Hence
$$\begin{aligned} A &= \sqrt{14 \times 8 \times 4 \times 2} \\ &= 8\sqrt{14} \\ &= 29.9, \text{ to 3 figures.} \end{aligned}$$

The area is 29.9 sq. in., assuming 3-figure accuracy in the data.

EXERCISES

For oral answers

ress as entire radicals:

1. $\sqrt{5} \times \sqrt{7}$.
2. $\sqrt{10} \times \sqrt{32}$.
3. $\sqrt{2} \times \sqrt{3} \times \sqrt{4}$.
4. $\sqrt{ax} \times \sqrt{by}$.
5. $\sqrt{5a} \times \sqrt{6b}$.
6. $\sqrt{11x} \times \sqrt{12y}$.
7. $2\sqrt{3}$.
8. $3\sqrt{7}$.
9. $5\sqrt{3}$.
10. $3\sqrt{11}$.
11. $8\sqrt{2}$.
12. $100\sqrt{6}$.
13. $a\sqrt{5}$.
14. $5\sqrt{a}$.
15. $a\sqrt{b}$.

ress as mixed radicals:

16. $\sqrt{32}$.
17. $\sqrt{50}$.
18. $\sqrt{80}$.
19. $\sqrt{45}$.
20. $\sqrt{24}$.
21. $\sqrt{700}$.
22. $\sqrt{162}$.
23. $\sqrt{125}$.
24. $\sqrt{72}$.

25. $\sqrt{4a^2b}$. 26. $\sqrt{8m^3}$. 27. $\sqrt{18r^5}$.
28. $\sqrt{a^3b^2c}$. 29. $\sqrt{x^4y^5}$. 30. $\sqrt{25x^7y^9}$.

For written answers

Express as entire radicals:

31. $\sqrt{14} \times \sqrt{17}$. 32. $6\sqrt{5}$. 33. $12\sqrt{12}$.
34. $3ab^2\sqrt{5}$. 35. $2\sqrt{3}\sqrt{xy^2}$. 36. $15x\sqrt{20y}$.

Express as mixed radicals:

37. $\sqrt{252}$. 38. $\sqrt{275}$. 39. $\sqrt{338}$.
40. $\sqrt{a^3b^5c^7}$. 41. $\sqrt{98x^2y^3}$. 42. $\sqrt{567mn^4}$.

(*a*) Express the following quantities as simply as possible means of radicals. (*b*) Assuming the data to have 3-figure curacy, find the answers correct to 3 figures.

43. The side of a square whose area is 108 sq. cm.

44. The radius of a circle whose area is 616 sq. cm.

45. The diagonal of a rectangle whose sides are 8 ft. and 1

46. The area of an isosceles triangle whose sides are 9, 9, 12 in., respectively.

47. The area of a triangle whose sides are 20, 32, and 44 respectively.

Express as simply as possible by means of radicals:

48. The diagonal of a square whose side is $3a$ in.

49. The area of an equilateral triangle whose side is $2b$ i

50. The diagonal of a rectangle whose sides are $2s$ ft. $4s$ ft.

51. The area of a triangle whose sides are $4k$, $6k$, and $8k$

Addition and Subtraction of Similar Radical

Since $$2x + 3x = 5x,$$
it follows, by substituting $\sqrt{2}$ for x, that
$$2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}.$$

e two radicals $2\sqrt{2}$ or $\sqrt{8}$ and $3\sqrt{2}$ or $\sqrt{18}$ are said to be
nilar or ***of the same kind.***

Similar radicals are added and subtracted like similar terms.

ample.

$$\sqrt{20} + \sqrt{45} - \sqrt{80} = \sqrt{4 \times 5} + \sqrt{9 \times 5} - \sqrt{16 \times 5}$$
$$= 2\sqrt{5} + 3\sqrt{5} - 4\sqrt{5}$$
$$= (2 + 3 - 4)\sqrt{5}$$
$$= \sqrt{5}.$$

The sum of two radicals which are not of the same kind, such $\sqrt{2} + \sqrt{3}$, cannot be expressed as a single radical term. To id errors, you should note carefully the distinction between $+ \sqrt{b}$ and $\sqrt{a + b}$. *These are not equal.* For example, $+ \sqrt{9} = 5$ and $\sqrt{4 + 9} = \sqrt{13} = 3.61$, approximately. ain, $\sqrt{2} + \sqrt{3} = 1.414 + 1.732 = 3.146$ and $\sqrt{2 + 3} =$ $= 2.236$, each to 4 figures.

EXERCISES

The first 6 may be done orally.

ress as simply as possible:

1. $5\sqrt{3} - 7\sqrt{3} + 4\sqrt{3}$.
2. $6\sqrt{7} - 4\sqrt{7} - \sqrt{7}$.
3. $5\sqrt{2} - 2\sqrt{8}$.
4. $9\sqrt{2} - \sqrt{72}$.
5. $\sqrt{20} - \sqrt{5}$.
6. $\sqrt{500} - \sqrt{125}$.
7. $\sqrt{12} - \sqrt{27} + \sqrt{48}$.
8. $\sqrt{50} - \sqrt{32} + \sqrt{18}$.
9. $\sqrt{640} - \sqrt{360} + \sqrt{160}$.
10. $\sqrt{72} + \sqrt{32} - \sqrt{98}$.
11. $\sqrt{175} + \sqrt{112} - \sqrt{252}$.
12. $\sqrt{288} - \sqrt{200} + \sqrt{128} - \sqrt{72}$.
13. $7a\sqrt{3} - 2\sqrt{12a^2} + \sqrt{27a^2} - a\sqrt{75}$.
14. $5\sqrt{44} - 3\sqrt{99} - 2\sqrt{176} + \sqrt{275}$.

15. Prove that $\sqrt{48} - \sqrt{27} = \sqrt{3}$ and verify by finding t square roots from the tables.

16. Prove that $\sqrt{125} - \sqrt{80} = \sqrt{5}$ and verify by worki out the square roots.

17. Use the tables to show the difference in value betwe $\sqrt{7} + \sqrt{3}$ and $\sqrt{10}$.

Multiplication of Radicals

While no new principles are involved in the multiplicati of radicals, practice is necessary in carrying out the steps.

Example 1. Find the product $6\sqrt{5} \times \frac{1}{2}\sqrt{5}$.

Since the factors can be taken in any order, we take t product of the rational factors and the product of the rational factors. This gives

$$6 \times \tfrac{1}{2} \times \sqrt{5} \times \sqrt{5} = 3 \times 5 = 15.$$

Example 2. Find the product $3\sqrt{2} \times 4\sqrt{7}$.

This product is $3 \times 4 \times \sqrt{2} \times \sqrt{7}$
or $12\sqrt{14}$.

We consider next the product of two binomials involv radicals. Such a product is found in the same manner as $(a + x)(b + y)$. It is advisable to write the product in horizontal line.

Example 3. $(2 + \sqrt{3})(5 + \sqrt{2}) = 10 + 2\sqrt{2} + 5\sqrt{3} + \sqrt{6}$

Example 4. $(3\sqrt{2} - 2\sqrt{3})^2 = 18 - 12\sqrt{6} + 12$
$= 30 - 12\sqrt{6}.$

Example 5.

$(2\sqrt{a} - 3\sqrt{b})(5\sqrt{a} + 6\sqrt{b}) = 10a + 12\sqrt{ab} - 15\sqrt{ab} - 18$
$= 10a - 3\sqrt{ab} - 18b.$

ample 6. $(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2}) = 5-\sqrt{10}+\sqrt{10}-2$
$= 3.$

On account of the simple result, this last example has par-ular interest. The two expressions which form this product e called ***conjugate radicals.*** More generally, $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$, where a and b are rational numbers, are conjugate dicals.

The product of two conjugate radicals is a rational number.
$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$

EXERCISES

The first 15 may be done orally.

d the values of the following products:

1. $5\sqrt{6}\times 8\sqrt{7}$.
2. $9\sqrt{2}\times\sqrt{8}$.
3. $6\sqrt{10}\times 4\sqrt{5}$.
4. $(3+\sqrt{5})(4+\sqrt{5})$.
5. $(5-\sqrt{3})(6+\sqrt{3})$.
6. $(3-2\sqrt{2})(4+3\sqrt{2})$.
7. $(5+\sqrt{15})(5-\sqrt{15})$.
8. $(3\sqrt{3}-5)(3\sqrt{3}+5)$.
9. $(\sqrt{8}+\sqrt{5})(\sqrt{8}-\sqrt{5})$.
10. $(7+2\sqrt{10})(7-2\sqrt{10})$.
11. $(3\sqrt{2}+2\sqrt{3})(3\sqrt{2}-2\sqrt{3})$.
12. $(5\sqrt{2}-7)(5\sqrt{2}+7)$.
13. $(\sqrt{5}+\sqrt{3})^2$.
14. $(2\sqrt{3}-1)^2$.
15. $(1+\sqrt{2}+\sqrt{3})^2$.
16. $\sqrt{50}\times\sqrt{125}$.
17. $\sqrt{27}\times\sqrt{48}$.
18. $\sqrt{10}\times\sqrt{1000}$.
19. $7\sqrt{2}\times 3\sqrt{5}\times 4\sqrt{3}$.
20. $\sqrt{2}(\sqrt{18}+\sqrt{32}+\sqrt{50})$.
21. $(\sqrt{11}+2\sqrt{3})^2$.
22. $(2\sqrt{7}-3\sqrt{5})^2$.
23. $(3\sqrt{6}+2\sqrt{10})^2$.
24. $(8\sqrt{5}+6\sqrt{3})(5\sqrt{5}-7\sqrt{3})$.
25. $3\sqrt{15}(7\sqrt{5}-4\sqrt{3})$.

26. $(2\sqrt{14} + 3\sqrt{6})(2\sqrt{14} - 3\sqrt{6})$.

27. $(\sqrt{x+1} + \sqrt{x-1})(\sqrt{x+1} - \sqrt{x-1})$.

28. $(\sqrt{45} + \sqrt{75} + \sqrt{5})(\sqrt{45} - \sqrt{75} + \sqrt{5})$.

29. $(1 + \sqrt{2})(1 + \sqrt{3})(\sqrt{2} + \sqrt{3})$.

30. $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$.

31. Any integer can be expressed as the product of two co jugate radicals in any number of ways. For example,

$$7 = 9 - 2 = (3 + \sqrt{2})(3 - \sqrt{2}).$$

Give another similar expression for 7. Give two similar e pressions for 10.

Find, in terms of radicals, the areas of the following figur Then, assuming 3-figure accuracy, evaluate the results by t use of tables.

32. A square with side $6 + \sqrt{3}$ in.

33. A rectangle with sides $10 - 2\sqrt{2}$ and $5 + 4\sqrt{2}$ in.

34. A triangle with base $\sqrt{10} + \sqrt{5}$ in. and height $3 + \sqrt{2}$

35. A circle with radius $7 - \sqrt{7}$ cm.

36. The square on the hypotenuse of a right triangle wh sides about the right angle are $2 + \sqrt{3}$ in. and $2 - \sqrt{3}$ in.

37. The outer curved surface of a tile with radius $2\sqrt{3}$ in. a height $8 + 3\sqrt{3}$ in.

Fractions with Radical Denominators

Since $$\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2},$$

it follows that $$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b},$$

and hence that $$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

The ratio of the square roots of two numbers is equal to the square root of the ratio of the numbers.

ample 1. $$\frac{\sqrt{30}}{\sqrt{3}} = \sqrt{\frac{30}{3}} = \sqrt{10}.$$

ample 2. $$\frac{\sqrt{15}}{\sqrt{2}} = \sqrt{\frac{15}{2}} = \sqrt{7.5}.$$

We obtain another way of expressing $\sqrt{15}/\sqrt{2}$ by mul-lying the numerator and the denominator by $\sqrt{2}$, an operation ich does not alter the value of the fraction. This gives

$$\frac{\sqrt{15}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{30}}{2} \text{ or } \tfrac{1}{2}\sqrt{30},$$

raction with a rational denominator. The advantage of the m $\frac{1}{2}\sqrt{30}$ over $\sqrt{7.5}$ is not great. It is just as easy to extract square root of 7.5 as it is of 30; on the other hand, if you are ng a table which gives the square roots of integers only, the m $\frac{1}{2}\sqrt{30}$ has an advantage over $\sqrt{7.5}$.

A problem of more interest arises when the denominator of a ction has more than one term. Consider the fraction

$$\frac{1}{\sqrt{2}+1}.$$

computational purposes this fraction would be simpler if its ominator were a rational number. How would you change fraction into an equal fraction with an integral denominator?

* * The suggestion of conjugate radicals gives the answer. ce

$$(\sqrt{2}+1)(\sqrt{2}-1) = 2 - 1 = 1,$$

it follows that the desired result will be obtained by multiplyi the numerator and the denominator by $\sqrt{2} - 1$. This gives

$$\frac{1}{\sqrt{2}+1} = \frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$$

$$= \frac{\sqrt{2}-1}{2-1}$$

$$= \sqrt{2} - 1.$$

This operation is called ***rationalizing the denominator.*** T foregoing example shows very well the advantage which resu from it. To compute the value of $\frac{1}{\sqrt{2}+1}$ in its present for we would write it $\frac{1}{1.414+1} = \frac{1}{2.414}$, which still requires operation of long division. On the other hand, after ration izing the denominator, we write at once

$$\sqrt{2} - 1 = 0.414.$$

Example 3. Express with rational denominators

$$(a) \quad \frac{1}{\sqrt{3}}, \quad (b) \quad \frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}.$$

$$(a) \quad \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

$$(b) \quad \frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}} = \frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}} \times \frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}} = \frac{7-2\sqrt{21}+}{7-3}$$

$$= \frac{10-2\sqrt{21}}{4} = \frac{5-\sqrt{21}}{2}.$$

In the fraction of Example 3(*b*), it is possible also to ratio ize the numerator. This is done by multiplying both numera

d denominator by $\sqrt{7} + \sqrt{3}$. For most purposes there is ater advantage in rationalizing the denominator than in ionalizing the numerator.

ample 4. Solve for x, expressing the root with a rational nominator, if any:

$$x\sqrt{3} = x\sqrt{2} + 5.$$

We have

$$x(\sqrt{3} - \sqrt{2}) = 5.$$

Therefore

$$x = \frac{5}{\sqrt{3} - \sqrt{2}}$$

$$= \frac{5}{\sqrt{3} - \sqrt{2}} \times \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}}$$

$$= \frac{5(\sqrt{3} + \sqrt{2})}{3 - 2} = 5(\sqrt{3} + \sqrt{2}).$$

EXERCISES

The first 9 may be done orally.

press in simpler form with rational denominators, if any:

1. $\sqrt{68} \div 2$.
2. $\sqrt{54} \div 3$.
3. $\sqrt{57} \div \sqrt{3}$.
4. $\sqrt{77} \div \sqrt{7}$.
5. $\sqrt{21} \div \sqrt{2}$.
6. $\sqrt{25} \div \sqrt{3}$.
7. $\dfrac{1}{\sqrt{5} - 2}$.
8. $\dfrac{1}{\sqrt{10} + 3}$.
9. $\dfrac{1}{4 - \sqrt{15}}$.
10. $\dfrac{\sqrt{56}}{\sqrt{12}}$.
11. $\dfrac{15}{2\sqrt{3}}$.
12. $\dfrac{2\sqrt{35}}{5\sqrt{14}}$.
13. $\dfrac{15}{4 - \sqrt{7}}$.
14. $\dfrac{2}{6 - \sqrt{5}}$.
15. $\dfrac{4\sqrt{2}}{\sqrt{15} + \sqrt{11}}$.

After rationalizing denominators, use the table to find the valu of the following, correct to 3 figures:

16. $\dfrac{2\sqrt{6}}{5\sqrt{2}-4\sqrt{3}}$. 17. $\dfrac{5+\sqrt{14}}{5-\sqrt{14}}$. 18. $\dfrac{\sqrt{11}+5}{\sqrt{11}-3}$.

19. $\dfrac{3\sqrt{7}-6}{2\sqrt{7}+5}$. 20. $\dfrac{5\sqrt{5}-3\sqrt{3}}{5\sqrt{5}+3\sqrt{3}}$. 21. $\dfrac{5\sqrt{2}-2\sqrt{5}}{7\sqrt{2}+3\sqrt{5}}$.

Express with rational numerators:

22. $4-\sqrt{15}$. 23. $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$. 24. $\dfrac{\sqrt{102}-10}{2}$.

Do the following problems without finding decimal equivale for the radicals:

25. Show that $\dfrac{1}{4-\sqrt{15}}$ is only slightly less than 8.

26. Show that $\dfrac{1}{\sqrt{10}-3}$ is only slightly greater than 6.

27. Find the nearest integer to $\dfrac{1}{6-\sqrt{35}}$. Is the inte greater or less?

28. Find the nearest integer to $\dfrac{1}{\sqrt{50}-7}$. Is the inte greater or less?

29. Find which is the greater of $\dfrac{1}{3-2\sqrt{2}}$ and $\dfrac{1}{5-2\sqrt{6}}$

30. Find which is the greater of $7-4\sqrt{3}$ and $8-3\sqrt{7}$

Solve the following equations for x, expressing each root w rational denominator, if any:

31. $x\sqrt{2}=\sqrt{3}$. 32. $x\sqrt{3}=5+x\sqrt{2}$.

33. $x\sqrt{5}=2-x\sqrt{3}$. 34. $2x+\sqrt{3}=x\sqrt{3}+2$.

35. $x\sqrt{5} - 2 = 2x + \sqrt{5}$.

36. $4x\sqrt{3} - 6\sqrt{2} = 5\sqrt{3} - 3x\sqrt{2}$.

lve for x and y, expressing the roots with rational denominators:

37. $x\sqrt{2} + y\sqrt{3} = 6.$
$x\sqrt{2} - y\sqrt{3} = 4.$

38. $x + y\sqrt{2} = 6.$
$x\sqrt{2} - y = 9.$

39. $x\sqrt{2} + y\sqrt{3} = 2.$
$x\sqrt{3} + y\sqrt{2} = 3.$

40. $x\sqrt{5} + 2y = 3\sqrt{15}.$
$x\sqrt{3} - y = 2\sqrt{15}.$

EXERCISES ON CHAPTER IX

A

1. What is a rational number?

2. Show that the arithmetic average of two rational numbers lso rational.

3. Of the following numbers, list the rational and irrational nbers separately and express the rational numbers without t signs:

$\sqrt{6}$, $\sqrt{3.6}$, $\sqrt{0.36}$, $\sqrt{625}$, $\sqrt{62.5}$, $\sqrt{6.25}$, $\sqrt{0.625}$, $\sqrt{0.0625}$.

4. Express as entire radicals: $7\sqrt{5}$, $6\sqrt{3}$, $r\sqrt{\pi}$, $ab^2\sqrt{c}$.

5. Express as mixed radicals: $\sqrt{176}$, $\sqrt{325}$, $\sqrt{8a^2b}$, $\sqrt{12ab^2c^3}$.

6. Find the value of $\sqrt{48} - \sqrt{108} + \sqrt{147} - \sqrt{75}$.

7. Find the value of $\sqrt{175} - \sqrt{112} + \sqrt{63} - \sqrt{28}$.

8. Simplify the following products: (*a*) $2\sqrt{3} \times 3\sqrt{2}$, $5\sqrt{8} \times 2\sqrt{2}$, (*c*) $7\sqrt{6} \times 2\sqrt{5}$, (*d*) $\sqrt{50} \times \sqrt{80}$, $\sqrt{32} \times \sqrt{48}$.

9. Simplify the following quotients: (*a*) $\frac{\sqrt{54}}{3}$, (*b*) $\frac{\sqrt{28}}{\sqrt{7}}$, $\frac{\sqrt{98}}{7\sqrt{2}}$, (*d*) $\frac{\sqrt{51}}{\sqrt{17}}$, (*e*) $\frac{\sqrt{44}}{\sqrt{99}}$.

0. Express in simple form: (*a*) $\sqrt{15} \times \sqrt{10} \times \sqrt{6}$, $\sqrt{21} \times \sqrt{30} \times \sqrt{70}$, (*c*) $\sqrt{39} \times \sqrt{21} \times \sqrt{91}$.

11. Find the value of (*a*) $(\sqrt{3}+2)(\sqrt{3}-2)$, (*b*) $(\sqrt{7}-2)(\sqrt{7}+2)$, (*c*) $(4-\sqrt{10})(4+\sqrt{10})$.

12. Write a multiplier of $\sqrt{21}+\sqrt{17}$ which will make t product a rational number and find the product.

13. Simplify $\dfrac{\sqrt{5}+2}{\sqrt{5}-2}$.

14. Express with rational denominators: (*a*) $\dfrac{9}{\sqrt{7}-2}$, (*b*) $\dfrac{8}{\sqrt{10}-\sqrt{6}}$.

15. Simplify $\dfrac{1}{\sqrt{3}+\sqrt{2}}+\dfrac{1}{\sqrt{3}-\sqrt{2}}$.

16. Expand (*a*) $(2+\sqrt{3})^2$, (*b*) $(3-\sqrt{2})^2$, (*c*) $(4+\sqrt{5})^2$.

17. Simplify the following products: (*a*) $(4-\sqrt{5})(2+\sqrt{}$, (*b*) $(\sqrt{7}-2)(\sqrt{7}+3)$, (*c*) $(5+\sqrt{11})(4-\sqrt{11})$.

18. Express in simpler form: (*a*) $(\sqrt{2})^5$, (*b*) $(\sqrt{3})^8$, (*c*) $(\sqrt[3]{}$, (*d*) $(\sqrt[5]{2})^{10}$, (*e*) $(\sqrt[4]{5})^2$.

B

19. Express as mixed radicals: (*a*) $\sqrt[3]{16}$, (*b*) $\sqrt[3]{54}$, (*c*) $\sqrt[3]{30}$, (*d*) $\sqrt[3]{24}$, (*e*) $\sqrt[3]{500}$.

20. Simplify:

(*a*) $\dfrac{2\sqrt{6}}{3\sqrt{2}}\times\dfrac{\sqrt{8}}{4\sqrt{3}}$, (*b*) $\dfrac{\sqrt{10}}{\sqrt{6}}\times\dfrac{\sqrt{18}}{\sqrt{15}}$, (*c*) $\dfrac{3\sqrt{15}}{2\sqrt{2}}\times\dfrac{4\sqrt{6}}{6\sqrt{5}}$.

21. Expand the following products: (*a*) $(a+\sqrt{b})(a-\sqrt{}$, (*b*) $(p+q\sqrt{r})(p-q\sqrt{r})$, (*c*) $(\sqrt{a}-x\sqrt{x})(\sqrt{a}+x\sqrt{x})$.

22. Expand the following products:

(*a*) $(\sqrt{a+b}+\sqrt{a})(\sqrt{a+b}-\sqrt{a})$,

(*b*) $(\sqrt{4a+3b}-2\sqrt{a})(\sqrt{4a+3b}+2\sqrt{a})$,

(*c*) $(\sqrt{2a+b}+\sqrt{2a-b})(\sqrt{2a+b}-\sqrt{2a-b})$.

23. Expand (*a*) $(2\sqrt{5}+3)^2$, (*b*) $(5-3\sqrt{2})^2$, (*c*) $(\sqrt{8}-\sqrt{6})^2$.

24. Expand
$(2\sqrt{7})+4)(2\sqrt{7}-4)$, (*b*) $(3\sqrt{10}+5\sqrt{2})(3\sqrt{10}-5\sqrt{2})$,
$(\sqrt{560}+\sqrt{550})(\sqrt{560}-\sqrt{550})$.

25. Show that $\dfrac{1}{5+\sqrt{3}}+\dfrac{1}{5-\sqrt{3}}$ is a rational number and
l it.

26. Show that $(2+\sqrt{3})^3+(2-\sqrt{3})^3$ is a rational number
l find it.

27. Find the value of $\dfrac{\sqrt{2}+1}{\sqrt{3}+1}-\dfrac{\sqrt{2}-1}{\sqrt{3}-1}$.

28. Simplify $\dfrac{(\sqrt{7}-2)(\sqrt{7}+3)}{(5-\sqrt{7})(\sqrt{7}+4)}$.

29. Find the value of $(2-\sqrt{2})(3-\sqrt{3})(2+\sqrt{2})(3+\sqrt{3})$.

30. Express 77 as the product of four expressions analogous
hose of Ex. 29.

31. Expand $(\sqrt{2}+\sqrt{3}+\sqrt{5})^2$.

32. Find the value of
$(1+\sqrt{2}+\sqrt{3})(1+\sqrt{2}-\sqrt{3})(1-\sqrt{2}+\sqrt{3})(1-\sqrt{2}-\sqrt{3})$.

33. Write four expressions whose product is an integer, given
t one of these expressions is $(\sqrt{5}+\sqrt{3}+\sqrt{2})$ and that
1 expression contains the same three roots. Find the product.

34. Expand and simplify $(1+\sqrt{2})^4$

35. Multiply $3\sqrt{7}-2\sqrt{10}+\sqrt{70}$ by $2-4\sqrt{7}+3\sqrt{10}$.

36. Rationalize the denominators of

$$(a)\ \frac{10\sqrt{3}-9\sqrt{2}}{5\sqrt{2}-4\sqrt{3}},\ (b)\ \frac{23\sqrt{5}-20\sqrt{7}}{5\sqrt{7}-6\sqrt{5}}.$$

37. Divide $24\sqrt{5}+31\sqrt{6}$ by $9\sqrt{5}+7\sqrt{6}$.

38. Simplify $\sqrt{50}-\sqrt{27}+\sqrt{128}-\sqrt{108}$ and, using the
es of $\sqrt{2}$ and $\sqrt{3}$, find its approximate value as a decimal.

39. Express as the product of two binomial expressions: (*a*) $\sqrt{30} + 2\sqrt{5} - \sqrt{6} - 2$, (*b*) $\sqrt{10} + \sqrt{14} + \sqrt{15} + \sqrt{}$

40. If $x = 1 + \sqrt{2}$, find the value of (*a*) $x - 1/x$, (*b*) $x^2 - 2x - 1$.

41. If $x = \dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$, find the value of $x + 1/x$.

42. Arrange in order of magnitude: $\sqrt{10} + \sqrt{8}$, $\sqrt{11} + \sqrt{}$ $\sqrt{12} + \sqrt{6}$. *Hint*: Compare their squares.

43. If a and b are positive numbers, which of $\sqrt{a} + \sqrt{b}$ $\sqrt{a + b}$ is the greater? Why?

44. Find the number of yards in the side of a square one-a plot of ground. Express the result as a simple radical evaluate to the nearest yard.

45. Find the radius of a circle whose area is 96π sq. ft.

46. An object falling from rest falls s feet in t seconds, wh $s = 16t^2$ very nearly. In what time will a stone fall 200 240 ft.?

47. If a body falls according to the law $s = 16t^2$, how does it take to fall (*a*) the first 64 feet, (*b*) the second 64 (*c*) the third 64 feet?

48. Find in acres, correct to a tenth of an acre, the area triangular field whose sides are 36, 40, and 56 rods, respecti

49. An equilateral triangle has the same area as a recta 8 inches by 12 inches. Find an expression for the length o side, in terms of radicals.

50. The height of an isosceles triangle is 3/2 of the base the equal sides are 20 cm. Find the base.

51. A rafter on a barn is 30 feet long and rises 2 feet for horizontal distance of 1 foot. How far does the rafter r (*a*) horizontally, (*b*) vertically?

52. Find the area of a triangle whose sides are 4, 5, and respectively.

C

53. A rectangular room has a floor space 15 feet by 12 feet and 9 feet high. Find (*a*) the length of the diagonal of the floor,) the length of the diagonal of the room from a corner of the or to the opposite corner of the ceiling. Express the answers mixed radicals.

54. Two sides of a parallelogram are 3 in. and 6 in. and the orter diagonal is 5 in. Find the area of the parallelogram.

55. The sum and the product of two rational numbers are ional. Is the sum of two irrational numbers necessarily ir- ional? Is the product necessarily irrational? Give examples illustrate your answers.

56. Give all the integral values of x, y, and z which will make $\sqrt{x} + \sqrt{y} + \sqrt{z}$ equal to (*a*) 0, (*b*) 1, (*c*) 2, (*d*) 3, (*e*) 4.

57. Express with rational denominator $\dfrac{1}{1 + \sqrt{5} - \sqrt{3}}$.

nt: Use two rationalizing steps.

58. Express with rational denominator $\dfrac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$.

59. Express with rational denominator $\dfrac{59}{\sqrt{7} + \sqrt{5} + \sqrt{3}}$.

60. By observing the expansion of $(\sqrt{a} + \sqrt{b})^2$, find bi- nial expressions for $\sqrt{4+2\sqrt{3}}$, $\sqrt{14+6\sqrt{5}}$, and $\sqrt{31+10\sqrt{6}}$.

61. By observing the expansion of $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$, find square root of $14 + 2\sqrt{30} - 2\sqrt{18} - 2\sqrt{15}$.

62. By observing the expansion of $(\sqrt{a} + \sqrt{b})^3$, find the e root of $17\sqrt{2} + 11\sqrt{5}$.

63. An integer can be expressed as the product of prime fac- s in only one way. (For example, $12 = 2 \times 2 \times 3$.) Is it e that an integer can be expressed as the product of expres- ns involving radicals in only one way? Give an example to strate your answer.

CHAPTER X

QUADRATIC EQUATIONS

Completing the Square

We have already met with many problems which require t solution of quadratic equations. These equations are of gr importance both in the applications of algebra and in buildi up the subject itself. It is necessary now to improve our sk in their solution.

The method which we have already used for solving quadra equations is that of factoring. This is generally the most c venient method provided the factors of the quadratic expressi involve only rational numbers. We proceed now to develo method by which we shall be able to solve any quadratic equati which has roots.

For an equation like $x^2 - 4 = 0$ let us recall that we m proceed in either of two ways, according to the following ste

Method 1.	**Method 2.**
If $x^2 - 4 = 0$,	If $x^2 - 4 = 0$,
then $(x - 2)(x + 2) = 0$	then $x^2 = 4$
whence $x - 2 = 0$ *or* $x + 2 = 0$	whence $x = \pm\sqrt{4}$
and $x = 2$ or -2.	$= \pm 2$.

Now consider the example

$$x^2 - 8x - 20 = 0.$$

The method of factoring gives

$$(x - 10)(x + 2) = 0$$

whence

$$x = 10 \text{ or } -2.$$

Although Method 2 does not apply to this equation in s present form, it gives a suggestion for the solution. We writ first in the form

$$x^2 - 8x = 20.$$

the left member were now a perfect square, the roots of the uation would be obtained by taking the square roots of both des. Although $x^2 - 8x$ is not a perfect square, it consists of the st two terms in the square of $x - 4$. Hence, if we add 16 to th sides of the equation we get

$$x^2 - 8x + 16 = 36$$
$$(x - 4)^2 = 36.$$

ow, taking square roots gives

$$x - 4 = \pm 6$$

ence $\quad x = 4 \pm 6,$

at is $\quad x = 10 \text{ or } -2.$

This solution depends on a device which is known as ***mpleting the square.*** The expression $x^2 - 8x$ became a mplete square on the addition of 16. In like manner $x^2 + 2ax$ comes a complete square on the addition of a^2. It should now a simple matter for you to complete the squares asked for in e following exercises. If these present any difficulty, consult e rule which follows the exercises.

EXERCISES

ld a term to each of the following in order to make the sum a mplete square:

1. $x^2 + 4x$.	**2.** $x^2 + 12x$.	**3.** $x^2 + 20x$.	**4.** $x^2 + 24x$.
5. $x^2 - 6x$.	**6.** $x^2 - 14x$.	**7.** $x^2 - 18x$.	**8.** $x^2 - 30x$.
9. $a^2 + 2ab$.	**10.** $x^2 - 8ax$.	**11.** $y^2 + 14by$.	
12. $z^2 - 6cz$.	**13.** $m^2 + m$.	**14.** $r^2 - 5r$.	
15. $t^2 + 9a^2t$.	**16.** $s^2 - 11b^2s$.	**17.** $p^2 - pq$.	

The method of completing a square can now be stated as a e. This rule is derived from the fact that the expression

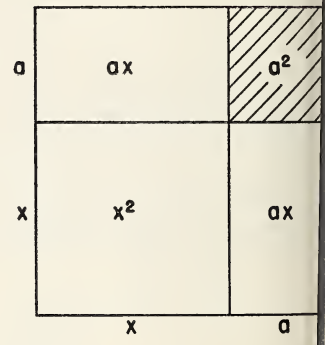
$$x^2 + 2ax$$

comes a complete square when a^2 is added. *When the co-cient of the square term is* 1, *the number which completes the*

square is ***the square of half the coefficient of the line term.***

> ***$x^2 + bx$ becomes a complete square by the addition of $\left(\frac{b}{2}\right)^2$.***

It is not hard to see how the method of completing the square suggested itself to the early mathematicians, who were accustomed to geometrical reasoning. You see that, in the figure, the area consisting of the square (x^2) and the two rectangles (each ax) requires the addition of the square (a^2) in order to make the whole square complete.



Solution of Quadratic Equations by Completing Squares

Example 1. By completing the square, solve $x^2 + 10x - 7 =$

The steps are as follows, with the explanations in parentheses.

If $x^2 + 10x - 7 = 0,$

then $x^2 + 10x = 7.$

(Add 7 to both sides so as to remove the constant from the left side.)

It follows that $x^2 + 10x + 5^2 = 7 + 25,$

(*Find the number which completes the square of the left side and add it to both sides.*)

which can be written $(x + 5)^2 = 32.$

(*Express the left side as a square.*)

Hence $$x + 5 = \pm\sqrt{32} = \pm 4\sqrt{2}$$

(*Take the square root of each side.*)

and $$x = -5 \pm 4\sqrt{2}.$$

Verification: Substitute $-5 + 4\sqrt{2}$ for x in $x^2 + 10x - 7.$

We get

$$(-5 + 4\sqrt{2})^2 + 10(-5 + 4\sqrt{2}) - 7$$
$$= 25 - 40\sqrt{2} + 32 - 50 + 40\sqrt{2} - 7$$

which reduces to 0. Similarly, the other root can be verified.

Notice that the two roots of this equation are the irrational nbers $-5 + 4\sqrt{2}$ and $-5 - 4\sqrt{2}$, of which one is positive l one is negative.

ample 2. Solve $3x^2 - 10x + 6 = 0.$

This equation differs from that of the preceding example in ving the coefficient of x^2 different from 1. By dividing both es by 3 we reduce this equation to the same type as Example 1.

From $$3x^2 - 10x + 6 = 0$$

we obtain $$x^2 - \frac{10}{3}x + 2 = 0$$

(*Divide both sides by the coefficient of* x^2.)

and hence $$x^2 - \frac{10}{3}x = -2.$$

(*Remove the constant term from the left side by subtracting* 2 *from each side.*)

It follows that $x^2 - \frac{10}{3}x + \left(\frac{5}{3}\right)^2 = -2 + \frac{25}{9}$,

(Find the number which completes the square of the left side and add it to both sides.)

which can be written $\left(x - \frac{5}{3}\right)^2 = \frac{7}{9}$.

(Express the left side as a square and collect terms on the right.)

Hence $x - \frac{5}{3} = \pm\sqrt{\frac{7}{9}} = \pm\frac{\sqrt{7}}{3}$

(Take the square root of each side.)

and $x = \frac{5}{3} \pm \frac{\sqrt{7}}{3}$ or $\frac{5 \pm \sqrt{7}}{3}$.

(Solve for x.)

Verification: Take $x = \frac{5 - \sqrt{7}}{3}$.

Then

$$3x^2 - 10x + 6 = 3\left(\frac{5 - \sqrt{7}}{3}\right)^2 - 10\left(\frac{5 - \sqrt{7}}{3}\right) + 6$$

$$= 3\,\frac{25 - 10\sqrt{7} + 7}{9} - \frac{50 - 10\sqrt{7}}{3} +$$

$$= \frac{25 - 10\sqrt{7} + 7 - 50 + 10\sqrt{7}}{3} + 6$$

$$= 0.$$

Similarly the other root can be verified. Both roots of equation are irrational and positive.

The steps in solving a quadratic equation by the method of completing the square.

(*Suppose the given equation is written with the right member* 0 *and with* x *for the unknown.*)

1) *Divide through by the coefficient of* x^2, *if it is different from* 1.
2) *Subtract from each side of the equation the term independent of* x.
3) *Add to each side the square of* $\frac{1}{2}$ *the coefficient of* x.
4) *Express the left side as a square and collect terms on the right.*
5) *Take the square root of each side.*
6) *Solve the resulting linear equations for* x.

EXERCISES

lve the following equations by the method of completing uares.

1. $x^2 + 2x - 15 = 0$.
2. $x^2 - 10x + 21 = 0$.
3. $x^2 + 12x + 35 = 0$.
4. $2x^2 + x - 15 = 0$.
5. $6x^2 - 5x - 21 = 0$.
6. $x^2 - x - 90 = 0$.
7. $x^2 - 2x - 21 = 0$.
8. $x^2 + 4x + 1 = 0$.
9. $x^2 - 14x + 25 = 0$.
10. $x^2 + 22x - 135 = 0$.
11. $10x^2 + 23x - 5 = 0$.
12. $x^2 + 6x + 2 = 0$.
13. $x^2 + 11x + 27 = 0$.
14. $5x^2 + 4x - 2 = 0$.
15. $9x^2 - 11x + 3 = 0$.
16. $14x^2 + 53x + 14 = 0$.

r the following four problems evaluate each root to one cimal place.

17. $4x^2 + 4x - 1 = 0$.
18. $4x^2 + 20x + 15 = 0$.
19. $9x^2 - 48x + 59 = 0$.
20. $9x^2 - 42x + 29 = 0$.

The General Quadratic Equation

From our study of particular examples, we observe that a quadratic equations can be written in the general form

$$ax^2 + bx + c = 0, \qquad (1$$

where a, b, and c are given numbers and x is the unknow

Thus, for example, the equation

$$(x - 3)^2 = 5 - (2x + 1)^2$$

becomes, on expansion,

$$x^2 - 6x + 9 = 5 - 4x^2 - 4x - 1$$

or

$$5x^2 - 2x + 5 = 0,$$

which is now in the general form (1) with $a = 5$, $b = -$ and $c = 5$.

EXERCISES

Reduce the following equations to the general form (1) a specify values of a, b, and c:

1. $7x = 4 - 2x^2$.
2. $3 + 4x^2 = 8x - 2x^2$.
3. $x(x - 4) = 3(x + 4)$.
4. $(x - 1)(2x + 1) = (x + 3)(x - 2)$.
5. $(3x - 2)^2 = (2x + 3)^2$.
6. $(x + 1)^3 = x(x^2 + x + 1)$.
7. $x(x + 1) - 3(x - 1) = 2(x + 3)^2$.
8. $x^2 = px + q$.
9. $(x - a)^2 = bx + c$.
10. $(mx + n)^2 = (px - q)^2$.
11. $(rx + 1)^2 = 2r(x - 1)$.
12. $(kx + h)^2 = h(h - kx)$.

The method of completing the square suffices to solve general quadratic equation, provided it has roots. We fol the steps outlined above.

om $ax^2 + bx + c = 0$

get $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{1}$$

d hence $$x^2 + \frac{b}{a}x = -\frac{c}{a}. \tag{2}$$

follows that $$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \tag{3}$$

ich can be written $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \tag{4}$$

nce $$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}, \tag{5}$$

d $$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \tag{6}$$

The roots of $ax^2 + bx + c = 0$ *are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is to be noted that the left member of equation (4), being a fect square, is either a positive number or zero. Hence if the ht member is a negative number, that is if $b^2 - 4ac$ is negative, s equation cannot hold and the given equation has no roots.

ample 1. Solve $25x^2 - 80x + 37 = 0$.

Comparing this equation with the general form, $ax^2 + bx + c = 0$, we have

$a = 25$, $b = -80$, $c = 37$. Hence

$$x = \frac{-(-80) \pm \sqrt{(-80)^2 - 4 \times 25 \times 37}}{2 \times 25}$$

$$= \frac{80 \pm \sqrt{6400 - 3700}}{50}$$

$$= \frac{80 \pm 10\sqrt{27}}{50}$$

$$= \frac{8 \pm 3\sqrt{3}}{5}.$$

Example 2. Solve $px^2 - 4qx + 4p = 0$.

This equation agrees with $ax^2 + bx + c = 0$ provid $a = p$, $b = -4q$, $c = 4p$. Hence

$$x = \frac{4q \pm \sqrt{16q^2 - 16p^2}}{2p}$$

$$= \frac{2q \pm 2\sqrt{q^2 - p^2}}{p}.$$

EXERCISES

1. State (orally) the procedure at each of the number steps in the solution of $ax^2 + bx + c = 0$ given above.

Solve the following equations by the use of the formula:

2. $x^2 - 9x + 14 = 0$.
3. $x^2 - 18x + 77 = 0$.
4. $x^2 + x - 30 = 0$.
5. $2x^2 + 5x - 12 = 0$.
6. $4x^2 - 4x - 35 = 0$.
7. $2x^2 - 19ax + 45a^2 = 0$.
8. $15x^2 - 2bx - b^2 = 0$.
9. $x^2 + 3x + 1 = 0$.
10. $x^2 - 4x - 2 = 0$.
11. $x^2 - 8x - 4 = 0$.
12. $x^2 + 10x + 13 = 0$.
13. $x^2 + 5x + 3 = 0$.
14. $6x^2 - 9x + 2 = 0$.
15. $72x^2 + 54x - 35 = 0$.
16. $9x^2 - 12x - 7 = 0$.
17. $x^2 + 16x - 1161 = 0$.
18. $9x^2 - 66x + 71 = 0$.
19. $6x^2 + 103x + 442 = 0$.

20. $30x^2 - 17x - 450 = 0$.
21. $32x^2 - 56x + 17 = 0$.
22. $x^2 + px + q = 0$.
23. $y^2 - 2my + n = 0$.
24. $at^2 + 4bt - 9c = 0$.
25. $Lm^2 + Rm + 1/C = 0$.
26. $3k^2z^2 + 7kz - 3 = 0$.
27. $300x - 100(x - 4) - 15x^2 = 0$.
28. $3(y + a) = cy^2$.
29. $3(z + 5)^2 = 20(z + 4)$.
30. $(x + 2)^3 - (x - 3)^3 = 50$.
31. $(r^2 + 3)^2 - (r^2 - 4)^2 = 7r + 3$.

The Graph of a Quadratic Function

The *equation* $ax^2 + bx + c = 0$ is satisfied only when the *pression* $ax^2 + bx + c$ has the value 0. Let us now consider e expression $ax^2 + bx + c$ by itself. It will assume different lues according to the values which are assigned to x. Such an pression is called a ***quadratic function*** of x.

As an example of a quadratic function we take

$$x^2 - 2x - 3.$$

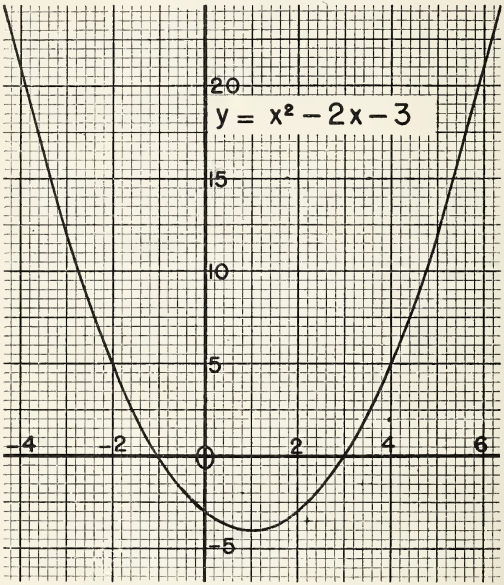
his is the particular case of $ax^2 + bx + c$ in which $a = 1$, $= -2$, and $c = -3$. We may denote the function by y so that

$$y = x^2 - 2x - 3.$$

he value of y depends upon that of x. What is one way to study is dependence? * * * Your previous experience tells you that e method is by means of a ***graph.*** We obtain data for drawing e graph by assigning values to x and calculating the cor- sponding values of y, thus obtaining the following table of lues:

x	-3	-2	-1	0	1	2	3	4	5	6
y	12	5	0	-3	-4	-3	0	5	12	21

When the points given by this table are plotted on squar paper and joined by a smooth curve, we obtain the graph sho in the figure. If now we think of x as growing continuously fr



large negative values to large positive values, then the gr portrays the life history of the function. It shows that, a increases from -3 to 1, the function decreases steadily to value -4, after which the function increases. The graph dicates also that as x departs from the value 1 in either direct the function behaves in the same way. This means that graph is symmetrical about a vertical line through $x = 1$.

A great deal of information is conveyed by the graph. points on the graph above the x-axis y is positive, for po

low the x-axis y is negative, and for points on the x-axis y is 0. r a given value of x we can tell the sign of a quadratic function erely by looking at its graph.

A quadratic function $ax^2 + bx + c$ is ***positive*** *for points on its graph which are* ***above*** *the axis of x; it is* ***negative*** *for points which are* ***below*** *the axis of x; it is zero at the points where the graph crosses the axis of x.*

The points at which the graph of $x^2 - 2x - 3$ crosses the s of x are seen to be those for which $x = -1$ and $x = 3$. nce these are the roots of the equation

$$x^2 - 2x - 3 = 0.$$

ese roots can also be found by the formula or by factoring. fact, the equation can be written

$$(x + 1)(x - 3) = 0$$

m which $$x = -1 \text{ or } 3.$$

The usefulness of this graph for solving equations is not con- ed to the single equation $x^2 - 2x - 3 = 0$. Let us imagine a e drawn parallel to the axis of x and one unit above it. At all nts on this line, $y = 1$, and hence at the points where the line sses the graph the values of x satisfy the equation

$$x^2 - 2x - 3 = 1$$
$$x^2 - 2x - 4 = 0.$$

e roots of this equation can therefore be read from the graph. ey are seen to be approximately 3.2 and -1.2.

With the foregoing example as a guide, how would you find n the graph the roots of

$$x^2 - 2x - 1 = 0?$$

This equation can be written in the form

$$x^2 - 2x - 3 = -2$$

or $y = -2$, where y belongs to the graph. The required roots a therefore the values of x at the points where the graph is cut the line for which $y = -2$.

It is now clear that by placing the edge of a sheet of pa or of a ruler parallel to the axis of x we can read the roots o great many equations of the form

$$x^2 - 2x + k = 0,$$

where k is any constant. If the ruler is transparent, it is eas to read the required coordinates.

Uses of the graph of a quadratic function, $ax^2 + bx + c$:

1) *It shows for what values of x the function is positive and for what values negative.*

2) *It shows the manner in which the function depends on x. In particular it shows:*

(*a*) *for what values of x the function is increasing and for what values decreasing;*

(*b*) *for what value of x the function is greatest or least;*

(*c*) *for what values of x the function changes rapidly and for what values slowly.*

3) *It shows whether the equation* $ax^2 + bx + k = 0$ *(where the c of the given function is replaced by any constant k) has roots and, if so, it solves this equation.*

EXERCISES

1. Use the graph on page 222 to answer the following:

(*a*) For what values of x is $x^2 - 2x - 3$ positive and what values negative?

(*b*) For what values of x does $x^2 - 2x - 3$ decrease a increases?

(*c*) What value of x makes $x^2 - 2x - 3$ least? What is this ast value?

(*d*) Does $x^2 - 2x - 3$ change more rapidly when x passes rough 2 or when x passes through 5?

om the same graph find the roots of:

2. $x^2 - 2x - 8 = 0$. 3. $x^2 - 2x = 0$.

4. $x^2 - 2x + 1 = 0$. 5. $x^2 - 2x - 1 = 0$.

6. What information does the graph give regarding the lution of $x^2 - 2x + 2 = 0$?

7. Draw the graph of $x^2 - 3x$. Does it appear to have an is of symmetry? What is the lowest point on the graph?

e the graph of Ex. 7 to find the roots of:

8. $x^2 - 3x = 0$. 9. $x^2 - 3x + 2 = 0$.

10. $x^2 - 3x - 4 = 0$. 11. $x^2 - 3x - 6 = 0$.

12. Draw the graph of $-x^2 + 7x - 6$. In what chief respect es it differ from the graphs previously drawn?

13. Describe the behaviour of the function $-x^2 + 7x - 6$ as ncreases from -1 to 7. For what values of x is the function sitive?

e the graph of Ex. 12 to solve:

14. $x^2 - 7x - 8 = 0$. 15. $x^2 - 7x + 3 = 0$.

16. Draw the graph of $x^2 - 4x + 4$. What peculiarity do u observe in this graph with respect to the axis of x?

e the graph of Ex. 16 to solve:

17. $x^2 - 4x + 4 = 0$. 18. $x^2 - 4x = 0$.

19. $x^2 - 4x - 4 = 0$.

20. Draw the graph of $x^2 + 4x + 5$. Describe the situation the graph with respect to the axis of x.

21. Does the function $x^2 + 4x + 5$ ever change sign? For at values of x does $x^2 + 4x + 5$ have its least value and what his least value?

e the graph of Ex. 20 to solve:

22. $x^2 + 4x + 3 = 0$. 23. $x^2 + 4x + 4 = 0$.

24. $x^2 + 4x - 5 = 0$.

25. Draw the graph of $x^2 + x - 6$.

From the graph of Ex. 25 find the roots of:

26. $x^2 + x - 2 = 0$. **27.** $x^2 + x - 12 = 0$.

28. Draw the graph of $-x^2 + 10x - 16$. What is the distan from the x-axis to the highest point on the graph?

29. Draw the graph of $x^2 + 5x$. If this graph were reflect in the axis of x, of what expression would the reflection be t graph?

30. Draw the graph of $-x^2 + 6x - 9$. Has the graph 'peak' or a 'valley'?

31. Show from the graph of Ex. 30 how the roots of t equation $-x^2 + 6x - 9 + k = 0$ change as k takes larger a larger values.

32. Draw the graph of $x^2 - 8x + 6$.

From the graph of Ex. 32 find the roots of:

33. $x^2 - 8x + 15 = 0$. **34.** $x^2 - 8x + 16 = 0$.

Problems Leading to Quadratic Equations

Now that we have general methods of solving quadra equations, we are able to solve any problem which leads to su an equation. A number of types of problems have already be considered in Chapter VII. Further examples are provided in t following list of exercises.

Among these problems we call attention to the followi types:

(1) Problems involving numbers;

(2) Geometrical problems involving lengths and areas;

(3) Problems involving prices. If n articles are priced p cents each, then the total cost is c cents, where $c = pn$;

(4) Problems involving time, rate, and distance, accordi to the formula $d = rt$.

Example 1. Two numbers differ by 2 and the sum of their squa is 202. Find the numbers.

If the smaller number is x, the larger is $x + 2$. Since the sum of their squares is 202, we have

$$x^2 + (x + 2)^2 = 202.$$

Hence $2x^2 + 4x + 4 = 202,$

from which $2x^2 + 4x - 198 = 0$

or $x^2 + 2x - 99 = 0.$

By factoring, we get $(x - 9)(x + 11) = 0.$

Therefore $x = 9$ or $-11.$

Hence the required numbers are 9 and 11 or -11 and -9. Both pairs of numbers are seen to satisfy the conditions of e problem.

ample 2. A's electric bill or January was \$8.40 and B's was 80. B's rate was $\frac{1}{4}$c. per kilowatt-hour less than A's and he d 40 k.w.h. more than A. Find A's rate.

Let A's rate be x cents per k.w.h., so that B's rate is $x - \frac{1}{4}$ cents per k.w.h. The number of k.w.h. used by A is $840/x$ and the number used by B is $880/(x - \frac{1}{4})$. Hence, by the conditions of the problem, we have

$$\frac{880}{x - \frac{1}{4}} - \frac{840}{x} = 40.$$

This gives $880x - 840(x - \frac{1}{4}) = 40x(x - \frac{1}{4})$

from which $4x + 21 = 4x^2 - x,$

or $4x^2 - 5x - 21 = 0.$

By factoring, we get $(x - 3)(4x + 7) = 0.$

Hence $x = 3$ or $-7/4.$

Since the rate cannot be negative, the root $-7/4$ is inadmissible. Hence the required rate is 3c. per k.w.h.

Verification: If A's rate is 3c. per k.w.h., the number of k.w.h. he used is 840/3 or 280. B's rate is then $2\frac{3}{4}$c. per k.w.h. and the number of k.w.h. he used is $880/2\frac{3}{4}$ or 320, which is 40 more than A used.

Example 3. Two trains make the same trip of 210 miles. T 'Express' goes on the average 16 m.p.h. faster than the 'Loc and covers the distance in $1\frac{1}{2}$ hours less time. Find the avera speeds of the two trains.

Let the average speed of the Local be v m.p.h. Then th of the Express is $v + 16$ m.p.h.

The time taken by the Local to go 210 miles is 21 hours and the time taken by the Express is $210/(v + 16)$ hou

Since the difference in these times is $1\frac{1}{2}$ hours, we have

$$\frac{210}{v} - \frac{210}{v + 16} = \frac{3}{2}.$$

Hence $$210(v + 16 - v) = \frac{3}{2}v(v + 16),$$

from which $$210 \times 16 \times 2 = 3(v^2 + 16v)$$

or $$v^2 + 16v - 70 \times 16 \times 2 = 0.$$

By factoring, we get $(v - 40)(v + 56) = 0$.

The only admissible root is $v = 40$. Hence the average sp of the Local is 40 m.p.h. and that of the Express is 56 m.p

Verification: At 40 m.p.h. the Local takes 210/40 or $5\frac{1}{4}$ hou At 56 m.p.h. the Express takes 210/56 or hours.

The difference in these times is $1\frac{1}{2}$ hours, as given in problem.

EXERCISES

1. Find two integers which differ by 3 and the sum of wh squares is 117.

2. The difference between the cubes of two consecut integers is 91. Find the integers.

3. A consignment of candy bars was bought for $50 and sold an advance of 1c. each. When the merchant had received 4 from sales, he had 100 bars left. Find the cost price per bar.

4. A merchant paid $150 for tennis racquets which he sold an advance of $5 each on the cost price. When he had sold all t two of the racquets, he had taken in $10 more than the cost ice. How many racquets did he buy?

5. Of three squares the side of the second is 1 foot longer an the side of the first and the side of the third is 1 foot longer an the side of the second. If the total area of the three squares 149 sq. ft., find the lengths of their sides.

6. A square enclosure is surrounded by a walk 4 feet wide, e area of the enclosure being equal to that of the walk. Find e side of the enclosure.

7. A fruit grower received $75.60 for a load of apples. When e price per bushel rose by 30c., he received the same amount 6 bushels less. Find the price per bushel.

8. A merchant paid $57.60 for a consignment of wheelrrows. When he had sold all but two of them at an advance $1.20 each on the cost price, he had taken in $2.40 more than had paid for the consignment. How many did he buy?

9. A car travelling 6 miles an hour faster than a second car kes 40 minutes less to cover a distance of 120 miles. Find the eeds of the two cars.

10. If, in a journey of 400 miles, the average speed were minished by 4 miles per hour, the time would be increased by hours. Find the average speed.

11. A line 10 inches long is divided into two parts on each of hich a square is constructed. If the sum of the areas of the two uares is $54\frac{1}{2}$ sq. in., find the lengths of the segments.

12. The cost of a number of articles is $7.20. Each article is rked at 4c. above cost. It is found that, when the original tlay is recovered, there is a margin of 15 articles left over. nd the number of articles bought.

13. An aeroplane and a ship leave at the same time on a rney of 600 miles. The plane travels 110 miles per hour faster

than the ship and arrives at the destination 11 hours earlier th the ship does. Find their speeds.

14. A stone is thrown vertically upward with a speed of 80 f per second. Assuming that its distance above the ground aft t seconds is $80t-16t^2$ feet, find in how many seconds it will 99 feet above the ground. Account for the two answers.

15. After the price of oranges rose 6c. a dozen, the number oranges I received for 18c. was 1 less than the number forme received for 22c. Find the original price per dozen.

16. Two river boats made trips of 84 and 96 miles respectivel The first travelled 2 miles per hour faster than the second a took 2 hours less for its trip. Find their speeds.

17. The force produced on the windshield of a car by a wi blowing with a velocity of v miles per hour is kv^2 pounds, wher is a constant. Find the velocity of the wind if an increase 6 miles per hour in the velocity would double the force on t windshield.

18. An eastbound car travelling at 20 miles per hour pas an intersection at 1 p.m. and a southbound car travelling 30 miles per hour passes the intersection at 2 p.m. At what ti are the cars 50 miles apart?

19. The sum of n of the integers 1, 2, 3, $\cdots$ is $\frac{1}{2}n(n +$ How many of these integers must be taken to yield a sum of 27

20. A man rows a boat 12 miles up stream and later rows ba to his starting point. If the stream has a current of 2 miles hour and he takes 4 hours longer for the trip upstream than the trip down, find his rowing speed in still water.

21. A fruit vendor took to market a truck-load of fruit which he paid \$78. He sold the fruit at an advance of 15c. basket. At the end of the day he had taken in \$88 and h 10 baskets unsold. How many baskets were in the original lo

22. A motorist made a journey of 200 miles and returned following day. On account of weather conditions his avera speed on the return journey was 8 miles per hour less than on first journey. His total travelling time was $11\frac{1}{4}$ hours. Find average speed on the first day's trip.

23. A belt which has a linear speed of 22 feet per second nnects two wheels. One wheel has a radius 6 in. less than that the other and it makes 6.3 more revolutions per second. Find e radii of the wheels. (Take $\pi = 22/7$).

24. A ball thrown from a point O at an angle of 45° with the rizontal describes a path whose graph is represented by $y = x - x^2/400$, where x is measured from O. If the ball on its downward urse strikes the roof of a barn at a height of 51 feet above the el of O, find the distance of the barn from O.

EXERCISES ON CHAPTER X

A

1. Give, by inspection, the roots of:
(*a*) $x^2 = 4$, (*b*) $x^2 = 4x$, (*c*) $4x^2 = x$, (*d*) $x^2 = 4x - 4$.

2. For each of the following, complete the square:
(*a*) $y^2 - 6y$, (*b*) $t^2 + 11t$, (*c*) $a^2b^2 - 8ab$, (*d*) $x^2 + 12xy$, $p^2 - 5pq$.

3. By factoring, solve:
(*a*) $6t^2 - t - 35 = 0$, (*b*) $15y^2 + 2y - 24 = 0$,
$4s^2 - 27s + 18 = 0$.

4. By completing squares, solve:
(*a*) $x^2 - 4x - 21 = 0$, (*b*) $y^2 - 3y - 40 = 0$,
$z^2 - 4z + 1 = 0$.

5. By using the formula, solve:
(*a*) $x^2 - 30x + 221 = 0$, (*b*) $z^2 - 6z - 7 = 0$,
$9m^2 - 9m - 10 = 0$.

6. Solve, by any method:
(*a*) $x^2 + 7x - 44 = 0$, (*b*) $2x^2 + 7x - 72 = 0$,
$6x^2 + 11x - 35 = 0$.

7. Simplify and solve:
(*a*) $(2x - 1)^2 = 3x^2 - 2$, (*b*) $(x + 4)^2 + 7 = (2x + 3)^2 - 5x$.

8. Solve:
(*a*) $x^2 + x - 110 = 0$, (*b*) $x^2 - 100x - 20{,}000 = 0$,
$12x = 5x^2$.

9. When the following expressions are reduced to the for $ax^2 + bx + c$, specify for each the values of a, b, and c:

(i) $(2x + 3)^2 - 3x(x - 1)$, (ii) $5(x - 2)^2 - 3(2x - 1)^2$; (iii) $(x + 2)^3 - x(x + 1)(x + 2)$.

10. Find two consecutive integers, the sum of whose squar is 85.

11. A line 1 foot long is divided into two parts on each which a square is constructed. The sum of the areas of the t squares is 74 sq. in. Find the lengths of the two parts of the li

12. The square of the sum of two consecutive integers ceeds the sum of their squares by 60. Find the integers.

13. One side of a rectangle is 7 feet shorter than the other a the area is 144 sq. ft. Find the dimensions.

14. When a number is increased by 5, the square of the n number exceeds by 25 twice the square of the first number. Fi the first number.

15. The perimeter of a rectangular table is 18 ft. and the a is 18 sq. ft. Find the dimensions.

16. The denominator of a fraction exceeds the numera by 1. A second fraction has terms respectively 1 less than th of the first. The first fraction exceeds the second by 1/30. Fi the first fraction.

17. Draw the graph of $x^2 - 4x$, for values of x from -3 t

18. From the graph of Ex. 17, tell for what values of x function is (*a*) positive, (*b*) negative, (*c*) zero, (*d*) increasi (*e*) decreasing.

19. From the graph of Ex. 17, tell for what value of x function is least and give the least value. What is the axis symmetry of the graph?

20. Use the graph of Ex. 17 to solve:

(*a*) $x^2 - 4x - 5 = 0$, (*b*) $x^2 - 4x - 12 = 0$, (*c*) $x^2 - 4x + 4 = 0$.

B

21. By completing squares, solve:

(*a*) $9y^2 - 6y - 7 = 0$, (*b*) $3r^2 - 4r - 2 = 0$,

$4z^2 + 8z - 5 = 0$.

22. Solve:
(*a*) $x^2 - mx + 2n = 0$, (*b*) $px^2 - qx - r = 0$,
$2ax^2 + 4bx + 3c = 0$.

23. Solve:
(*a*) $2x^2 - 7ax + 4a^2 = 0$, (*b*) $6x^2 - (10a - 9)x - 15a = 0$,
$a^2x^2 - abx - b^2 = 0$.

24. Find the ratio x/y, given that:
(*a*) $x^2 - 4y^2 = 0$, (*b*) $x^2 - 21xy + 108y^2 = 0$,
$x^2 - 6xy + 6y^2 = 0$.

25. Solve $\dfrac{x-3}{x+2} + \dfrac{x+4}{x-2} = \dfrac{5}{2}$.

26. Find four roots of $(x + 2/x)^2 = 9$.

27. Find four roots of $x^4 - 34x^2 + 225 = 0$.

28. Find all the roots of $(x^2 + x)^2 - 14(x^2 + x) + 24 = 0$.

29. Two numbers differ by 4 and their reciprocals differ by
0. Find the numbers.

30. A man travelled 20 miles at uniform speed. If his speed
been 1 mile per hour greater, his time would have been
our less. Find his speed.

31. Two cars travel towards each other from two towns 101
es apart. The second car leaves one hour after the first,
rels 2 miles per hour more slowly, and covers 36 miles before
meet. Find the speeds of the two cars, assuming the speeds
e greater than 10 miles per hour.

32. A pupil spent 60c. for notebooks. If they had been priced
c. less he could have bought one more notebook for the same
ay. How many did he buy?

33. A merchant paid $144 for shoes of a certain kind and the
e amount for shoes of another kind. The second kind cost
pair more than the first and he bought 8 pairs more of the
than of the second. Find the price per pair of each kind.

34. One side of a rectangular 12-acre field is 8 rods longer
the other. Find the sides.

35. Two rectangles have the same area, 96 sq. in. The short side of the first rectangle is 2 inches longer than the shorter si of the second; the longer side of the first is 4 inches shorter th the longer side of the second. Find the dimensions of the re angles.

36. Two cars start from the same town on the same highwa The second car leaves 15 minutes after the first, travels on t average 4 miles per hour faster and overtakes the first 110 mi from their starting point. Find the average speeds of t two cars.

37. Two ships travel at 20 knots on two straight paths wh cross at right angles. One ship arrives at the intersection of th paths 3 hours after the other. At what time later will the sh be 100 miles apart? (Give the answer to a tenth of an hour.)

38. Show that any two of the following equations have a r in common: (*a*) $2x^2 + 9x - 18 = 0$, (*b*) $6x^2 - 5x - 6 =$ (*c*) $3x^2 + 20x + 12 = 0$.

39. Write three quadratic equations each of which has root 5 and no two of which have another root in common.

40. Divide the number 2 into two parts such that their prod is equal to the difference of their squares. How many answ has the problem?

41. An aviator flying at a height of h miles above the ea has an uninterrupted view to the horizon for d miles, wher

$$d^2 = h^2 + 8000h.$$

Find the height to which an aviator should climb in order to a distance of 90 miles.

42. Draw the graph of $x^2 - 5x - 6$. Has this graph a 'pe or a 'valley'? What is its axis of symmetry?

43. From the graph of Ex. 42 find the roots of (*a*) $x^2 -$ $+ 6 = 0$, (*b*) $x^2 - 5x - 2 = 0$.

44. Draw the graph of $x^2 + 9x + 14$. Are there points this graph for which $y = 100$? Are there points for wh $y = -100$?

45. If $x^2 + 9x + 14 = k$ has roots, show from the graph Ex. 44 the effect on these roots of increasing k.

46. **In what distance can a driver stop his car?** Highway engineers have found that on a dry road a car with good brakes travelling at r miles per hour requires at least d feet for stopping, where

$$d = 0.045\, r^2 + 1.1r.$$

Find the maximum speed at which one could drive his car in order to be able to bring it to a stop in a distance of 100 feet.

47. **An engineering formula.** Engineers use a formula for the compression produced in a spring by a load dropped on it from a given height. For a certain type of spring the formula is

$$F^2 = 2P(s + F),$$

where F is the number of inches of compression, P the number of pounds in the load, and s the number of inches the load is dropped. Find the compression produced in the spring by a load of 3 pounds dropped from a height of 12 inches.

48. Using the formula of Ex. 47, find the compression produced in the spring by dropping a load of 4 pounds from a height of 5 inches.

49. In taking square roots of $x^2 = a^2$ you may write $\pm x = \pm a$. Show that this gives four possible arrangements of sign but that these reduce to the two arrangements of $x = \pm a$.

50. Draw the graph of $x^2 - 6x + 5$ for values of x from 0 to 9.

51. From the graph of Ex. 50, tell for what values of x the function is (*a*) positive, (*b*) negative, (*c*) zero, (*d*) greater than 5, (*e*) less than -3.

52. Use the graph of Ex. 50 to solve:
(*a*) $x^2 - 6x - 7 = 0$, (*b*) $x^2 - 6x + 9 = 0$, (*c*) $x^2 - 6x + 4 = 0$.

C

53. If r is a root of $ax^2 + bx + a = 0$, show that $1/r$ is also a root.

54. If r is a root of $x^2 - x + c = 0$, show that $1 - r$ is also a root.

55. Solve $\dfrac{x}{2(x+3)} + \dfrac{5}{3} = \dfrac{x^2}{9-x^2} + \dfrac{7x-2}{4(x-3)}$.

56. Given that $y = x^2 + 3x + 1$ and that $2y^2 - 5y - 3 =$ find all the possible values of x.

57. Draw the graph of $y = -2x^2 - 5x - 2$. Write t equations for which $y = 1$, $y = -2$, and $y = -5$ and solve th approximately from the graph.

58. A circle is inscribed in a square whose side is 10 inch A point on the circle is 2 in. from one side of the square. Find distances from the other three sides.

59. A carriage wheel 5 feet in diameter is blocked by a re angular block 6 in. high. Find the horizontal distance of t block from the centre of the wheel.

60. Carry out the following steps for solving the quadra equation $x^2 - 3x - 1 = 0$. Write the equation $x^2 = 3x +$ Draw carefully the graph of $y = x^2$ on a fairly large scale. the same figure draw the graph of $y = 3x + 1$ (a straight lin At the points where the two graphs meet the two y's are eq and hence $x^2 = 3x + 1$. The required roots are therefore values of x at the points where the two graphs meet.

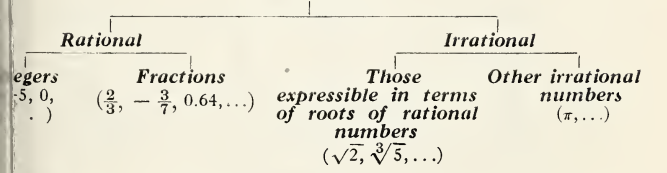
61. Using the graph of $y = x^2$ which you constructed Ex. 60, solve (*a*) $x^2 + x - 1 = 0$, (*b*) $x^2 - 2x - 2 = 0$. Obt the roots to one decimal place.

CHAPTER XI

JADRATIC EQUATIONS; FURTHER STUDY

A Classification of Numbers

The numbers which we have used so far have been classified ***rational*** or ***irrational.*** Of these the rational numbers consist ***integers and fractions.*** Of the irrational numbers, some are pressible by means of roots and some are not. All rational and ational numbers are designated as ***real numbers.*** These tements are embodied in the following classification:



Although this classification is sufficient for our present poses, those who make a more advanced study of algebra l find that an extension of the number system is made to lude other numbers—non-real or imaginary, as they are led.

Graphical Discussion of Roots

Let us return now to the general quadratic function

$$ax^2 + bx + c.$$

call the graphs which you have drawn for some particular mples of this function. These graphs are all graceful curves ch either rise indefinitely from a central valley or descend

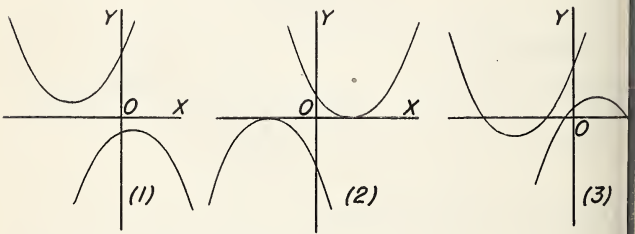
indefinitely from a central hill. They are all symmetrical abo a line perpendicular to the axis of x.

We may denote the quadratic function under considerati by y, so that

$$y = ax^2 + bx + c.$$

In the relation of the graph of this function to the axis of x, th are three possibilities:

(1) the graph may lie wholly above or wholly below the axis of as in Fig. 1.
(2) the graph may touch the axis of x, as in Fig. 2,
(3) the graph may cut the axis of x, as illustrated by either the curves of Fig. 3,



Suppose that the graph of $y = ax^2 + bx + c$ resembles eit of the curves of Fig. 3, so that the graph cuts the axis of x two distinct points. This means that y is equal to 0 for t distinct values of x or that the equation

$$ax^2 + bx + c = 0$$

has ***two distinct real roots.***

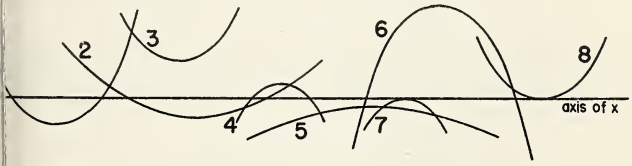
Again, if the graph resembles either of the curves of Fig then y is equal to 0 for only one value of x. Hence the equat $ax^2 + bx + c = 0$ has only one root. Notice, however, tha very slight change could be made in the curve which would m it cut the axis in two points close together. If such a change w made, the resulting equation would have two roots very nea

ual. For this reason the equation corresponding to either of the
aphs of Fig. 2 is said to have ***two equal roots.***

Finally, if the graph resembles either of those of Fig. 1, then y
never zero for any value of x. Hence, there are no real numbers
ich satisfy the equation. This is sometimes expressed by saying
at the equation has ***no real roots.***

The equation $\boldsymbol{ax^2 + bx + c = 0}$ *has two distinct real roots, two equal real roots, or no real roots, according as the graph of* $\boldsymbol{ax^2 + bx + c}$ *meets the x-axis in two points, one point, or no points, respectively.*

EXERCISE



If each of the curves in the figure is the graph of a function of
form $ax^2 + bx + c$, specify those for which the equation
$+ bx + c = 0$ has (a) real and distinct roots, (b) real and
al roots, (c) no real roots.

Algebraic Discussion of Roots

The possibilities in the character of the roots of

$$ax^2 + bx + c = 0$$

clearly shown by the possible graphs of $ax^2 + bx + c$. With
se possibilities in mind, we turn now to an algebraic dis-
sion of the roots. In this discussion, the coefficients a, b, and c
be understood to be real numbers.

We begin by writing down the equation and its roots.

> ***If*** $\quad ax^2 + bx + c = 0,$
>
> ***then*** $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

Under what condition does the equation have two distinct r
roots?

* * *

This condition is, clearly, that $b^2 - 4ac$ should be a posit
number. For, if $b^2 - 4ac$ is positive, then it has two real squ
roots, $\pm \sqrt{b^2 - 4ac}$, and hence the equation has two disti
real roots, namely,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We shall presently see that under no other condition are the ro
real and distinct. The relation of this condition to the graph f
$ax^2 + bx + c$ is now clear.

> *The geometrical meaning of the condition* $b^2 - 4ac > 0$ *is that the graph of* $y = ax^2 + bx + c$ *crosses the axis of x in two distinct points.*

Under what condition are the roots of the equation equa

* * *

The condition is that $b^2 - 4ac = 0$. For, then, $\sqrt{b^2 - 4ac}$ 0
and the two roots given by the formula both reduce to $-b/$

> *The condition* $b^2 - 4ac = 0$, *means, geometrically, that the gra*
> *of* $y = ax^2 + bx + c$ *touches the axis of x without crossing*

ally, the condition $b^2 - 4ac < 0$ means that the equation has real root. For, if $b^2 - 4ac$ is a negative number, it can have real number for its square root. (Since any real number must positive, negative, or zero, therefore the square of any real nber must be positive or zero.)

he condition $b^2 - 4ac < 0$ *means, geometrically, that the graph* $y = ax^2 + bx + c$ *lies wholly on one side of the axis of* x.

The character of the roots of the quadratic equation $+ bx + c = 0$ depends on the coefficients a, b, and c. More cisely, we see that *the expression whose value determines the racter of the roots is* $b^2 - 4ac$. This is called the ***discriminant*** he quadratic equation or of the quadratic function. The ılts of the preceding discussion can now be tabulated in the)wing form.

Value of the discriminant	*Roots of* $ax^2 + bx + c = 0$	*Graph of* $ax^2 + bx + c$
$^2 - 4ac > 0$	*Real and different*	*Crosses the axis of* x *at two points*
$^2 - 4ac = 0$	*Real and equal*	*Touches the axis of* x
$^2 - 4ac < 0$	*No real roots*	*Lies wholly on one side of the axis of* x

mple 1. Find, from the discriminant, the character of the s of (*a*) $2x^2 - 9x + 8 = 0$, (*b*) $9x^2 - 60x + 100 = 0$, $3x^2 + 8x + 6 = 0$.

a) Here $a = 2$, $b = -9$, and $c = 8$. Hence, the discriminant

$$b^2 - 4ac = (-9)^2 - 4 \times 2 \times 8 = 81 - 64 = 17.$$

Since this is positive, the roots of the equation are real ınd different.

(*b*) $a = 9$, $b = -60$, and $c = 100$. Hence
$b^2 - 4ac = (-60)^2 - 4 \times 9 \times 100 = 3600 - 3600 = 0$
It follows that the roots of the equation are real and equ

(*c*) $a = 3$, $b = 8$, and $c = 6$. Hence
$b^2 - 4ac = 8^2 - 4 \times 3 \times 6 = 64 - 72 = -8$.
Since the discriminant is negative, the equation has no real roots.

Example 2. For what values of m does the equation $2x^2 - 3x + m = 0$ have (*a*) equal roots, (*b*) real and unequ roots?

The discriminant is $(-3)^2 - 4 \times 2 \times m = 9 - 8m$.

(*a*) The roots are equal when the discriminant is 0, i.e., w
$8m = 9$ or when $m = 9/8$.

(*b*) The roots are real and unequal when the discrimin is positive, i.e., when
$8m < 9$ or when $m < 9/8$.

EXERCISES

Form the discriminant of each of the following functions:

1. $x^2 - 5x - 2$.
2. $2x^2 + 3x + 2$.
3. $4x^2 - 24x + 36$.
4. $3x^2 - 4x + 6$.
5. $7x^2 + 9x + 3$.
6. $2x^2 + 2\sqrt{6}x + 3$.
7. From your answers to the preceding exercises tell, r each, how the graph of the quadratic function is related to e axis of x.

Without solving the following equations, find the character of e roots of each:

8. $x^2 + 7x + 1 = 0$.
9. $3x^2 + 5 = 0$.
10. $49x^2 - 42x + 9 = 0$.
11. $5x^2 + 9x + 4 = 0$.
12. $8x^2 + 4\sqrt{10}x + 5 = 0$.
13. $4x^2 + 13x + 11 = 0$.
14. $0.1x^2 + 1.2x + 3 = 0$.
15. $5x^2 + 7 = 8(3x - 4)$.

16. $(t + 2)^2 - 3(t - 1) - 4 = 0.$
17. $(y + 3)^2 = 5(2y + 3).$
18. $0.09z^2 - 0.66z + 1.21 = 0.$
19. $ax^2 + bx - a = 0.$

d for what values of a the following equations have real roots:

20. $x^2 + 8x + a = 0.$
21. $x^2 + ax + 4 = 0.$
22. $ax^2 + 2x + 1 = 0.$
23. $x^2 + 2x + a^2 = 0.$

d for what values of m the following equations have equal ts:

24. $x^2 - 12x + m = 0.$
25. $8x^2 + mx + 2 = 0.$
26. $mx^2 + 3x + 5 = 0.$
27. $mx^2 - 5x + m = 0.$

The Sum and the Product of the Roots

Let us, for brevity, denote the two roots of $ax^2 + bx + c = 0$ m and n.

$$m = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$n = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

en the two roots are added, the radical terms disappear and have

$$m + n = -\frac{b}{a}.$$

v form the product of the two roots. We get

$$mn = \left(-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{b^2}{4a^2} - \frac{b^2-4ac}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}.$$

If m and n are the roots of $ax^2 + bx + c = 0$, then

$$m + n = -\frac{b}{a},$$

$$mn = \frac{c}{a}.$$

If the terms of the given equation are divided by a, it becon

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Since $\frac{b}{a} = -(m + n)$ and $\frac{c}{a} = mn$, the equation can be writ

$$x^2 - (m + n)\,x + mn = 0.$$

This result admits a ready verification. For, in its present fo the left member can be factored, giving

$$(x - m)(x - n) = 0,$$

from which the roots are seen to be m and n. The follow conclusion can now be stated:

If the sum of the roots of a quadratic equation is denoted by S and the product of the roots by P, the equation can be written

$$x^2 - Sx + P = 0,$$

or, in full,

$x^2 -$ (*sum of roots*)$x +$ *product of roots* $= 0$.

nce, if the roots of an equation are given, the equation can be tten down immediately.

ample. Construct the equation whose roots are -3 and $7/4$.

First solution, *by constructing factors*:

The required equation is

$$(x + 3)(x - 7/4) = 0,$$

which reduces to $$x^2 + \frac{5}{4}x - \frac{21}{4} = 0,$$

or $$4x^2 + 5x - 21 = 0.$$

Second solution, *by using the sum and product of the roots*: Denote the sum and the product of the roots by S and P, respectively. Then

$$S = -3 + 7/4 = -5/4,$$

and $$P = -3 \times 7/4 = -21/4.$$

Hence the required equation,

$$x^2 - Sx + P = 0,$$

becomes $$x^2 - (-5/4)x + (-21/4) = 0,$$

or $$4x^2 + 5x - 21 = 0,$$

as before.

EXERCISES

l the sum and the product of the roots of:

. $9x^2 - 15x + 5 = 0$. 2. $3x^2 - 11 = 0$.

3. $5y^2 + 10y - 16 = 0$. 4. $0.1r^2 - 2.34r - 3.2 = 0$.

5. $x^2 + px + q = 0$. 6. $6z^2 + mz - 3n = 0$.

n the equations whose roots are:

7. -7 and 3. 8. $6a$ and $8a$. 9. $-\frac{1}{2}c$ and $\frac{3}{4}c$.

10. $a + b$ and $a - b$. 11. $2r$ and $3s$.

12. 3.1 and -1.3. **13.** $2 + \sqrt{3}$ and $2 - \sqrt{3}$.
14. $\frac{1}{2}(5 + \sqrt{2})$ and $\frac{1}{2}(5 - \sqrt{2})$.
15. $a + \sqrt{b}$ and $a - \sqrt{b}$.
16. Test whether 10 and -3.6 are the roots of $x^2 + 6.4x - 36 = 0$.
17. Test whether -2.5 and 8 are the roots of $2x^2 - 11x - 40 = 0$.
18. If 3 is a root of $x^2 - 5x + k = 0$, find (*a*) the other ro (*b*) the value of k.
19. If -7 is a root of $x^2 + px - 28 = 0$, find (*a*) the ot root, (*b*) the value of p.
20. Supply the missing term in $x^2 - 12x \cdot \cdot \cdot = 0$, given t one root is 5.
21. Supply the missing term in $x^2 \cdot \cdot \cdot - 18 = 0$, given t one root is 6.
22. Find the roots and supply the missing term in $x^2 - 9x \cdot \cdot \cdot = 0$, given that one root is twice the other.

Symmetric Functions of the Roots

The expressions $m + n$ and mn are unaltered in value w m and n are interchanged. For this reason they are ca ***symmetric functions*** of m and n. All other expressions w the same property are also called symmetric functions. Exam of symmetric functions of m and n are $m^2 + n^2$, $mn^2 + n$, $m/n + n/m$, $m^3 + n^3 - m - n$. When a quadratic equatio given, it is possible to find the value of a given symmetric fu tion of its roots. The following example illustrates the meth

Example 1. If m and n are the roots of $2x^2 - 5x - 4 = 0$, the value of (*a*) $1/m + 1/n$, (*b*) $m^2 + n^2$, (*c*) $m/n + n$, (*d*) $m^3 + n^3$.

(*a*) From the given equation we have

$$m + n = 5/2$$

and

$$mn = -2.$$

Hence
$$\frac{1}{m}+\frac{1}{n}=\frac{n+m}{mn}$$
$$=\frac{5/2}{-2}=-\frac{5}{4}.$$

(*b*) Our object is to express m^2+n^2 in terms of $m+n$ and mn. We have

$$\begin{aligned} m^2+n^2 &= (m+n)^2-2mn \\ &= (5/2)^2-2(-2) \\ &= 25/4+4=41/4. \end{aligned}$$

(*c*) By adding the fractions and using the result of (*b*), we get
$m/n+n/m=(m^2+n^2)/mn$
$$=\frac{41/4}{-2}=-41/8.$$

(*d*) Two methods may be suggested here. The first makes use of factoring. Thus,

$$\begin{aligned} m^3+n^3 &= (m+n)(m^2-mn+n^2) \\ &= (m+n)\{(m+n)^2-3mn\} \\ &= \tfrac{5}{2}\{(\tfrac{5}{2})^2-3(-2)\} \\ &= \tfrac{5}{2}(\tfrac{25}{4}+6) \\ &= \tfrac{5}{2}(\tfrac{49}{4})=\tfrac{245}{8}. \end{aligned}$$

A second method makes use of $(m+n)^3$. We have

$$\begin{aligned} m^3+n^3 &= (m+n)^3-3m^2n-3mn^2 \\ &= (m+n)^3-3mn(m+n) \\ &= (5/2)^3-3(-2)(5/2) \\ &= 125/8+15=30\tfrac{5}{8}. \end{aligned}$$

EXERCISES

m and n are the roots of $5x^2-8x-3=0$, find the value of:

1. m^2+n^2.
2. $1/m+1/n$.
3. $m/n+n/m$.

If m and n are the roots of $8x^2 + 9x + 2 = 0$, find the value o

4. $1/m^2 + 1/n^2$. 5. $m^3 + n^3$. 6. $m^3n^2 + m^2n^3$.

For the equation $3x^2 + 5x - 3 = 0$, find the value of:

7. The sum of the reciprocals of the roots.
8. The sum of the squares of the reciprocals of the roots.
9. The sum of the cubes of the reciprocals of the roots.

If m and n are the roots of $2x^2 - 5x - 1 = 0$, find the value o

10. $m^2/n + n^2/m$. 11. $(m - n)^2$.
12. $m^3 + m^2n + mn^2 + n^3$.

Construction of Equations

A problem which frequently arises is that of constructing equation whose roots are related in a given manner to those o given equation. One method of solution is to find the roots of t given equation and use them to find the roots of the requir equation. This method is convenient if the given equation h integral roots which are easily found. Suppose, for example, wish to find the equation whose roots are respectively tw those of

$$x^2 - 5x + 6 = 0.$$

The roots of the given equation are seen to be 2 and 3. Her those of the required equation are 4 and 6 and the requi equation is

$$(x - 4)(x - 6) = 0$$

or

$$x^2 - 10x + 24 = 0.$$

For most equations, however, such problems are more eas solved by making use of the formulas for the sum and produc the roots. This method and an alternative method which is of useful are illustrated in the following examples.

Example 1. Form the equation whose roots are, respectiv 3 times the roots of $3x^2 + 4x - 5 = 0$.

Method 1. Denote by m and n the roots of $3x^2 + 4x - 5 = 0$.

Then $$m + n = -4/3$$
and $$mn = -5/3.$$

The roots of the required equation are $3m$ and $3n$. The sum S and the product P of these roots are found as follows:

$$S = 3m + 3n = 3(m + n) = 3(-4/3) = -4;$$
$$P = 3m \times 3n = 9mn = 9(-5/3) = -15.$$

The required equation is

$$x^2 - Sx + P = 0$$

or, after substitution, $x^2 + 4x - 15 = 0$.

Method 2. Denote the unknown in the required equation by y. Then y is related to the x which satisfies $3x^2 + 4x - 5 = 0$ by the relation

$$y = 3x.$$

Hence $$x = y/3$$

and, by substituting $y/3$ for x in the given equation, we obtain the required equation which must be satisfied by y:

$$3(y/3)^2 + 4(y/3) - 5 = 0.$$

On multiplying by 3, we get

$$y^2 + 4y - 15 = 0,$$

which is the same equation as before, with the unknown denoted by y.

mple 2. Form the equation whose roots are the squares of se of $x^2 + px + q = 0$.

Method 1. Denote by m and n the roots of $x^2 + px + q = 0$.

Then $$m + n = -p$$
and $$mn = q.$$

The roots of the required equation are m^2 and n^2. We next orm their sum S and product P.

$$S = m^2 + n^2 = (m + n)^2 - 2mn = (-p)^2 - 2q = p^2 - 2$$
$$P = m^2n^2 = (mn)^2 = q^2.$$

The required equation is now

$$x^2 - Sx + P = 0$$

or

$$x^2 - (p^2 - 2q)x + q^2 = 0.$$

Method 2. Denote the unknown in the required equati by y. Then y is related to the x which satisfies $x^2 + px + q = 0$ by the relation $y = x^2$. Hence $x = \pm\sqrt{y}$ a by substituting $\pm\sqrt{y}$ for x in the given equation, we get t required equation which must be satisfied by y:

$$(\pm\sqrt{y})^2 + p(\pm\sqrt{y}) + q = 0,$$

or

$$y \pm p\sqrt{y} + q = 0.$$

This is now reduced to the standard quadratic form isolating the square root term on one side and then squari both sides. This gives

$$y + q = \mp p\sqrt{y},$$

from which

$$(y + q)^2 = p^2y,$$

or

$$y^2 + (2q - p^2)y + q^2 = 0.$$

Except for the letter which denotes the unknown, this is same equation as obtained before.

EXERCISES

1. Form the equation whose roots are, respectively, twic great as those of $2x^2 - 6x - 3 = 0$.

2. Form the equation whose roots are, respectively, 5 ti as great as those of $x^2 - x - 1 = 0$.

3. Form the equation whose roots are, respectively, squares of those of $3x^2 + 4x - 2 = 0$.

4. Form the equation whose roots are, respectively, 2 gre than those of $5x^2 + 8x + 2 = 0$.

5. Form the equation whose roots are, respectively, 3 less
ın those of $2x^2 + 9x + 6 = 0$.
6. Form the equation whose roots are, respectively, the
)es of those of $x^2 + 6x + 4 = 0$.
If m and n are the roots of $2x^2 + 4x - 5 = 0$, find the
ıation whose roots are:
7. $m + n$ and mn. 8. $1/m$ and $1/n$. 9. m^3 and n^3.
If m and n are the roots of $ax^2 + bx + c = 0$, find the
ıation whose roots are:
10. $m + 1$ and $n + 1$. 11. $3m$ and $3n$.
12. m^2 and n^2.

The Factors of a Quadratic Function

We have seen that the method of factoring is useful in finding
roots of quadratic equations. The simplest method of solving

$$6x^2 - 7x - 20 = 0$$

o factor the left member, obtaining

$$(2x - 5)(3x + 4) = 0,$$

n which $$x = 5/2 \text{ or } -4/3.$$

If the factors of a quadratic function are known, the roots of
corresponding equation are known. Thus, the roots of

$$(x - 2 - \sqrt{6})(x - 2 + \sqrt{6}) = 0$$

$$x = 2 \pm \sqrt{6}.$$

s equation, with the factors multiplied out, is

$$(x - 2)^2 - 6 = 0,$$
$$x^2 - 4x - 2 = 0.$$

'e were asked to factor the trinomial $x^2 - 4x - 2$, we might
e some difficulty in guessing the factors. However, from the
in which the equation has just been formed, we know that

$$x^2 - 4x - 2 = (x - 2 - \sqrt{6})(x - 2 + \sqrt{6}).$$

This example suggests that it is possible to make further u of the connection between a quadratic function and a quadra equation. This connection is made clear by considering t general quadratic ***function***

$$ax^2 + bx + c$$

and the corresponding ***equation***

$$ax^2 + bx + c = 0.$$

If the roots of the ***equation*** are m and n, then $m + n = -b$ and $mn = c/a$. Then the quadratic ***function*** is

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) \\ &= a\{x^2 - (m + n)x + mn\} \\ &= a(x - m)(x - n). \end{aligned}$$

The two-fold connection is now clear and may be stated follows:

I. *Knowing the* ***factors of the quadratic function,*** *we a able to write the* ***roots of the corresponding equation.***
II. *Knowing the* ***roots of the quadratic equation,*** *we a able to write the* ***factors of the corresponding function.***

Since we have learned a formula for the roots of a quadratic equation, we have at hand a method of finding factors of any quadratic function. We shall restrict our cussion to those quadratic equations which have real ro The equation $x^2 + 1 = 0$ is a simple example of one whic satisfied by no real number and, correspondingly, the funct $x^2 + 1$ has no linear factors involving only real numbers.

The connection, as well as the distinction, between a adratic equation and a quadratic function are brought out in following table:

The quadratic equation	*The quadratic function*
$ax^2 + bx + c = 0$	$ax^2 + bx + c$
oots: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	*Factors:* $a(x - m)(x - n)$ *where* m *and* n *are the roots of* $ax^2 + bx + c = 0$
arning: *A quadratic function can have any number of values. It is not necessarily zero.*	

mple 1. Factor $2x^2 + 9x - 315$.

This function is equal to

$$2(x - m)(x - n)$$

where m and n are the roots of

$$2x^2 + 9x - 315 = 0.$$

These roots are $\dfrac{-9 \pm \sqrt{81 + 2520}}{4}$.

Simplifying, we get

$$\frac{-9 \pm \sqrt{2601}}{4} = \frac{-9 \pm 51}{4} = -15 \text{ or } 21/2.$$

Hence the required factors are

$$2(x + 15)(x - 21/2).$$

mple 2. Factor $4x^2 - 28x + 39$.

The roots of $4x^2 - 28x + 39 = 0$

are $\dfrac{28 \pm \sqrt{28^2 - 16 \times 39}}{8}$.

Simplifying, we get

$$\frac{28 \pm 4\sqrt{7^2 - 39}}{8} = \frac{7 \pm \sqrt{10}}{2}.$$

Hence the required factors are

$$4\left(x - \frac{7 + \sqrt{10}}{2}\right)\left(x - \frac{7 - \sqrt{10}}{2}\right)$$

which are more conveniently written as

$$(2x - 7 - \sqrt{10})(2x - 7 + \sqrt{10}).$$

It should be pointed out that it is possible to carry out t factoring of a quadratic function without reference to the c responding equation. This is done by the method of ***completi the square,*** as in the following solution.

Example 3. Factor $5x^2 - 8x - 3$.

We have

$$\begin{aligned} 5x^2 - 8x - 3 &= 5(x^2 - \tfrac{8}{5}x - \tfrac{3}{5}) \\ &= 5\{x^2 - \tfrac{8}{5}x + (\tfrac{4}{5})^2 - \tfrac{16}{25} - \tfrac{3}{5}\} \\ &= 5\{(x - \tfrac{4}{5})^2 - \tfrac{31}{25}\} \\ &= 5\left(x - \tfrac{4}{5} + \frac{\sqrt{31}}{5}\right)\left(x - \tfrac{4}{5} - \frac{\sqrt{31}}{5}\right. \end{aligned}$$

EXERCISES

Find linear factors of the following:

1. $x^2 - 5$.
2. $2x^2 - 3$.
3. $x^2 + 2x - 143$.
4. $6x^2 - x - 40$.
5. $16y^2 - 72y + 81$.
6. $y^2 - 6y + 4$.
7. $z^2 + 10z + 17$.
8. $z^2 - 8z - 2$.
9. $2t^2 - 6t + 1$.
10. $4t^2 - 16t + 5$.
11. $25w^2 + 60w + 8$.
12. $9a^2w^2 - 60abw + 100b^2$

13. $x^2 + px + q$. 14. $x^2 + ax - a^2$.

ow that the following have no real linear factors:

15. $4x^2 + 9$. 16. $x^2 + x + 1$.

17. $3y^2 + 9y + 7$. 18. $2z^2 + 10z + 13$.

EXERCISES ON CHAPTER XI

A

1. Write the classification of numbers, as on page 237, and ign each of the following numbers to its proper place: -8, 1.4, 0, $\sqrt{8}$, $-\sqrt{16}$, $\sqrt[3]{6}$, π^2, $\sqrt[4]{2}$, $\sqrt[3]{27}$, $-(1 + \sqrt{2})$.

2. What is the character of the roots of $ax^2 + bx + c = 0$ en the graph of $y = ax^2 + bx + c$ (*a*) cuts the axis of x in) points, (*b*) lies wholly above the axis of x, (*c*) touches the s of x?

3. Form the discriminant of each of the following functions l hence tell how its graph is related to the axis of x: $x^2 + 7x + 11$, (*b*) $2x^2 + 3x + 2$, (*c*) $4x^2 - 20x + 25$.

4. Find the sum and the product of the roots of $x^2 - 5 = 0$, (*b*) $x^2 + 6x = 0$, (*c*) $x^2 + x - 1 = 0$, $x^2 - px - q = 0$.

5. Find the sum and the product of the roots of $2x^2 - 5x + 3 = 0$, (*b*) $3x^2 - 8x + 4 = 0$, $12x^2 + 9x - 8 = 0$.

6. Form the equation for which the sum of the roots is 7/2 the product is 5/2.

7. Form the equation whose roots are (*a*) -6 and 9, (*b*) $3a$ $5a$, (*c*) $r + s$ and $r - s$.

8. Without solving the following equations, determine ther they have real roots and, if so, whether the roots are al or different: (*a*) $2x^2 - 5x + 3 = 0$, (*b*) $6y^2 - 8y + 3 = 0$, $16z^2 - 88z + 121 = 0$.

9. Determine, as in Ex. 8, the character of the roots of: $5x^2 + 7x + 3 = 0$, (*b*) $49r^2 + 42r + 9 = 0$, $7t^2 - 12t - 4 = 0$.

10. Supply the missing term in: (*a*) $x^2 - 7x + \cdots =$ given that one root is 4; (*b*) $x^2 + 5x + \cdots = 0$, given that o root is -4; (*c*) $x^2 + 10x - \cdots = 0$, given that one root is

11. Supply the missing coefficient in (*a*) $x^2 + \cdots x +$ $= 0$, given that one root is 3; (*b*) $x^2 - \cdots x - 24 = 0$, giv that one root is 8; (*c*) $x^2 + \cdots x - 30 = 0$, given that o root is -5.

12. Find the factors of (*a*) $x^2 - 3$, (*b*) $x^2 + 5x - 84$, (*c*) $x^2 - 9x - 90$.

13. Draw the graph of $x^2 - 3x$ for values of x from -3 to Does the graph open upward or downward? Locate its axis symmetry.

14. Use the graph you have drawn for Ex. 13 to find the roo if any, of (*a*) $x^2 - 3x = 0$, (*b*) $x^2 - 3x = 4$, (*c*) $x^2 - 3x =$ (*d*) $x^2 - 3x = -1$, (*e*) $x^2 - 3x = -3$.

15. Without drawing them, find out whether the graphs the following functions cut the axis of x, touch this axis, or wholly on one side of it:

(*a*) $4x^2 + 1$, (*b*) $4x^2 - 1$, (*c*) $100x^2 + 30x + 1$, (*d*) $-49x^2 + 70x - 25$.

16. If m and n are the roots of $x^2 + 5x + 3 = 0$, find t value of:

(*a*) $(m + n)^2$, (*b*) $m^2 + n^2$, (*c*) $(m - n)^2$, (*d*) $1/m + 1$ (*e*) $1/m^2 + 1/n^2$.

B

17. Given that $2x^2 - ax + 8 = 0$ has equal roots, find t value of a.

18. Given that $6x^2 - 5x + k = 0$ has equal roots, find t value of k.

19. If m and n are the roots of $x^2 - x + 7 = 0$, find t value of : (*a*) $m^2 + n^2$, (*b*) $m^2 + 5mn + n^2$, (*c*) $1/m + 1$ (*d*) $m^3 + n^3$.

20. If m and n are the roots of $x^2 - 5x + 2 = 0$, find t value of: (*a*) $m^2/n + n^2/m$, (*b*) $m^4 + n^4$.

21. Form the equation whose roots are, respectively, twice as at as those of $x^2 - 2x - 3 = 0$.
22. Form the equation whose roots are, respectively, 3 times se of $2x^2 + 5x - 4 = 0$.
23. Form the equation whose roots are, respectively, greater 2 than those of $x^2 - 8x + 4 = 0$.
24. Form the equation whose roots are, respectively, greater 3 than those of $x^2 + 9x + 6 = 0$.
25. Form the equation whose roots are, respectively, the ares of those of $x^2 - 2x - 3 = 0$.
26. Form the equation whose roots are, respectively, the ares of those of $3x^2 - 8x - 9 = 0$.
27. Form the equation whose roots are, respectively, the procals of those of $5x^2 + 6x - 4 = 0$.
28. Form the equation whose roots are, respectively, the ares of the reciprocals of those of $x^2 - 8x + 6 = 0$.
29. If m and n are the roots of $x^2 - 3x + 1 = 0$, form the ation whose roots are $4m + 1$ and $4n + 1$.
30. Form the equation whose roots are (*a*) $7 + \sqrt{11}$ and $\sqrt{11}$, (*b*) $5 + 2\sqrt{3}$ and $5 - 2\sqrt{3}$, (*c*) $\frac{7 + 3\sqrt{2}}{2}$ and $\frac{-3\sqrt{2}}{2}$.

31. Find for what values of p the equation $x^2 + px + 4 = 0$ (*a*) real and distinct roots, (*b*) real and equal roots, (*c*) no roots.
32. Find for what values of q the equation $x^2 + 2x + q = 0$ (*a*) real roots, (*b*) no real roots, (*c*) equal roots.
33. Find the factors of the following functions. Do the ors contain irrational numbers? (*a*) $10x^2 + 59x - 120$, $18x^2 - 45x - 50$, (*c*) $9x^2 - 42x + 49$.
34. Find the factors of the following functions. Do the ors contain irrational numbers? (*a*) $x^2 - 4x - 1$, $2x^2 + 10x + 9$, (*c*) $3x^2 + 12x + 10$.

35. For what values of b does the graph of $x^2 + bx +$ (*a*) cut the axis of x in two points, (*b*) touch the axis of x, (*c*) ha no point in common with the axis of x?

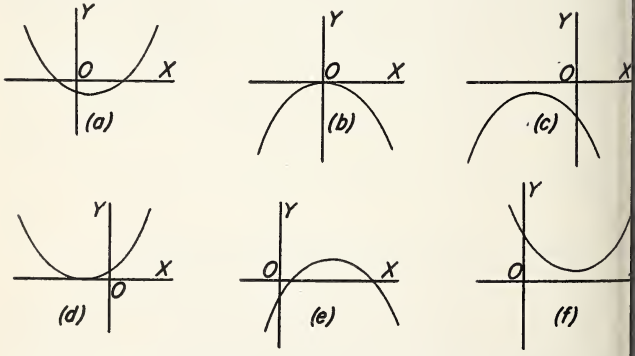
36. For each of the following functions, find the value of in order that the graph of the function may touch the axis of (*a*) $x^2 + px + 25$, (*b*) $x^2 + 8x + p$, (*c*) $px^2 + 6x + 4$, (*d*) $px^2 + 4x + p$, (*e*) $x^2 + px + p$.

37. Find an expression for the difference of the roots $ax^2 + bx + c = 0$.

38. Show that the roots of $ax^2 + bx + a = 0$ are reciproc of each other.

39. Find the condition under which $ax^2 + bx + c$ is the squa of a linear function. (*Hint*: Two equal factors imply two eq roots of the corresponding equation.)

40. Each of the accompanying graphs represents a functi $ax^2 + bx + c$.



For each case state the character of the roots of the equati $ax^2 + bx + c = 0$.

C

41. For each of the following functions, answer the questi below: (*a*) $25x^2 - 40x + 16$, (*b*) $-25x^2 + 40x - 18$, (*c*) $-25x^2 + 40x - 12$, (*d*) $25x^2 - 40x - 12$.

(i) Does the graph open upward or downward?
(ii) Does the graph meet the axis of x in no point, in one point, in two points?
(iii) If the graph cuts the axis of x in two points, is the origin tween these points?

42. Given that $px^2 + 4x + 3 = 0$ has real roots, find the eatest possible value of p.

43. Find the equation whose roots are, respectively, k times ose of $ax^2 + bx + c = 0$.

44. Find the equation whose roots, respectively, exceed those $ax^2 + bx + c = 0$ by k.

45. If a and c have opposite signs show that $ax^2 + bx + c = 0$ s real roots, of which one is positive and one is negative.

46. Give the signs of p and q in order that the roots of $x^2 + + q = 0$ may be (a) both positive, (b) both negative, (c) of posite signs, the numerically greater being positive, (d) of posite signs, the numerically greater being negative.

47. If a, b, and c are rational numbers, under what condition e the roots of $ax^2 + bx + c = 0$ (a) rational numbers, (b) ir- tional numbers?

48. Using the relation $ax^2 + bx + c = x^2(a + b/x + c/x^2)$, ove that, *when x is very large*, (a) b/x and c/x^2 are very small,) $a + b/x + c/x^2$ has the same sign as a. Hence show that, x gets very large positively or negatively, the graph of $^2 + bx + c$ opens upward if a is positive and downward if a is gative.

CHAPTER XII

SYSTEMS OF EQUATIONS

How Many Unknowns?

Suppose you were given the problem: *A man and his so together earn* \$65 *a week. How much does each earn?* * * You would say at once: "The problem is impossible. You ar asking me to find two numbers and you give me only one bit o information to find them with." If we add that the man make \$15 a week more than his son, then we have enough informatio to solve the problem. For then, if the man makes \$$x$ and the so \$$y$ a week, we have

$$x + y = 65$$
$$x - y = 15.$$

An inspection of these equations shows that $x = 40$ and $y = 2$

Take another problem: *The perimeter of a rectangular roo is* 42 *feet. Find its sides.* Again you say: "The information insufficient to solve the problem. In order to find two numbers must have two bits of information or two *data* (Lat. *datum* plural, *data*; something given) to use." Let us add to our in formation the fact that the area of the room is 108 square fee Then, if the two sides of the room are x feet and y feet respectiv ly, the data of the problem give

$$2x + 2y = 42$$
$$xy = 108.$$

Try to finish the problem for yourself. We shall return to it late

From these and similar problems we draw the followi conclusion:

> ***In order to find two unknown numbers, we must be given two equations or be able to find two equations from our data.***

Here is a third problem: *The sum of the ages of John, Dave, nd Tom is* 41 *years. Tom is* 3 *years older than John. Find their ges.*

It is clear that this problem, which asks for three ages but ves only two data, is also impossible. If we add that Tom's and hn's ages together are 13 years more than Dave's, then we ve a possible problem. If the numbers of years in John's, ave's, and Tom's ages, respectively, are x, y, and z, then the ree facts of the problem give rise to the three equations:

$$\begin{aligned} x + y + z &= 41 \\ z - x &= 3 \\ x + z - y &= 13. \end{aligned}$$

y to solve these equations for yourself.

In order to find three unknown numbers, we must be given three equations or be able to find three equations from our data.

Two Linear Equations in Two Unknowns

The solution of systems of two linear equations in two unowns was discussed in Chapter I. The method consists in *minating one of the unknowns.* The following example shows w this method of elimination can sometimes be extended to stems which are not linear in x and y. The equations of this mple are linear in the two unknowns $1/x$ and $1/y$.

ample. Solve:
$$\frac{15}{x} - \frac{8}{y} = 7 \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{25}{x} + \frac{6}{y} = 2 \quad . \quad . \quad . \quad . \quad (2)$$

To eliminate y, we multiply (1) by 3 and (2) by 4. This give

$$\frac{45}{x} - \frac{24}{y} = 21 \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{100}{x} + \frac{24}{y} = 8 \quad . \quad . \quad . \quad . \quad (4)$$

Add (3) and (4): $\frac{145}{x} = 29.$

Hence $29x = 145$

and $x = 5.$

Substitute 5 for x in (1). This gives

$$3 - \frac{8}{y} = 7$$

from which $\frac{8}{y} = 3 - 7 = -4$

and $y = -2.$

Answer: $x = 5$, $y = -2$.

These values are easily verified by substitution in the origin equations.

Now test your skill on the following exercises, referring agai if necessary, to the examples on pages 8 and 9.

EXERCISES

Solve the following systems of equations:

1. $4x - 5y = 13$
   $2x + 3y = 23.$

2. $4x + 5y = 6$
   $6x + 7y = 12.$

3. $8x + 3y =$
   $4x + 15y = 1$

4. $12x + 10y = 1$
   $8x - 15y = 18.$

5. $\frac{5}{x} - \frac{3}{y} = 2$; $\frac{3}{x} - \frac{1}{y} = 6.$

6. $\frac{3}{x}+\frac{2}{y}=-1;\quad \frac{7}{x}+\frac{4}{y}=3.$

7. $\frac{24}{x}-\frac{27}{y}=5$

$\frac{36}{x}-\frac{45}{y}=8.$

8. $\frac{3}{x}+\frac{4}{y}=\frac{30}{x}-\frac{32}{y}=1.$

9. $2x+3y=0$
$3x+4y=7xy.$

10. $\frac{2x}{5}+\frac{14}{y}=3$

$\frac{2x}{5}-\frac{14}{y}=5.$

11. $\frac{2}{x-y}+\frac{17}{x+y}=36$

$\frac{3}{x-y}-\frac{34}{x+y}=12.$

12. $x^2+y^2=25$
$2x^2-y^2=2.$

Three Linear Equations in Three Unknowns

In order to solve three linear equations in three unknowns, ır object is to obtain from them two equations in two unknowns. his requires us to ***eliminate one of the unknowns be-veen two pairs of the given equations.*** The solution of e following example shows the necessary steps.

xample 1. Solve:
$$3x+2y+5z=1 \quad . \quad . \quad . \quad (1)$$
$$7x-3y+z=2 \quad . \quad . \quad . \quad (2)$$
$$6x-5y-2z=-1 \quad . \quad . \quad . \quad (3)$$

Let us eliminate z between (1) and (2), then eliminate z between (2) and (3), thus obtaining two equations in the two unknowns x and y. The details are as follows:

Multiply both sides of (2) by 5 and subtract the resulting equation from (1). This gives

$$\begin{array}{rr} 3x+2y+5z= & 1 \\ 35x-15y+5z= & 10 \\ \hline -32x+17y= & -9 \end{array} \quad . \quad . \quad . \quad (4)$$

Multiply both sides of (2) by 2 and add the resulting equation to (3). This gives

$$\begin{aligned} 14x - 6y + 2z &= 4 \\ 6x - 5y - 2z &= -1 \\ \hline 20x - 11y &= 3 \quad . \quad . \quad . \quad (5) \end{aligned}$$

From (4) and (5) we have now to eliminate x or y. Attention to the equations shows that it is easier to eliminate x, which is done by multiplying both sides of (4) by 5 and both sides of (5) by 8 and then adding the resulting equations. We get

$$\begin{aligned} -160x + 85y &= -45 \\ 160x - 88y &= 24 \\ \hline -3y &= -21 \quad . \quad . \quad . \quad (6) \end{aligned}$$

Hence $y = 7.$

Now x is found by substituting 7 for y in (4) or in (5). We notice that the work is easier if we use (5). Hence

$$20x - 77 = 3$$

whence $20x = 80$

and $x = 4.$

It remains to find z. This is done by substituting 4 for x and 7 for y in (1) or in (2) or in (3). We choose (2). Then

$$28 - 21 + z = 2$$

whence $z = 2 - 28 + 21 = -5.$

Answer: $x = 4, y = 7, z = 5.$

Verification: Substitute these values in the left side of (1). This gives $12 + 14 - 25$ which equals 1, and agrees with the number on the right side of (1). Show that these values also satisfy (2) and (3).

Note on this solution. From the given set (1), (2) and (3) of three equations in three unknowns x, y, and z, we obtained by eliminating z a set (4) and (5) of two equations in two unknowns x and y and from these, by eliminating x, we obtained a single equation (6) in y.

3 equations 3 unknowns	→	*2 equations 2 unknowns*	→	*1 equation 1 unknown*

In solving three equations in three unknowns several steps are required. It is therefore important to work carefully and systematically. The following suggestions may be helpful.

(1) *If one of the three equations contains only two unknowns, begin by eliminating the third unknown from the other two equations.*

(2) *Study the equations to see which unknown can be most readily eliminated from two pairs of the given equations.*

(3) *In eliminating an unknown from a pair of equations observe whether it is possible to do so by multiplying only one of the equations and then adding or subtracting.*

Example 2. Solve:
$$\frac{6}{x} + \frac{15}{y} = 4 \quad . \quad . \quad . \quad (1)$$
$$\frac{18}{y} - \frac{16}{z} = 1 \quad . \quad . \quad . \quad (2)$$
$$\frac{14}{x} + \frac{12}{z} = 5 \quad . \quad . \quad . \quad (3)$$

These are linear equations in the three unknowns $1/x$, $1/y$, and $1/z$. The first equation contains only x and y. We can obtain a second equation in x and y by eliminating z between (2) and (3). The steps are as follows.

Multiply (2) by 3:
$$\frac{54}{y} - \frac{48}{z} = 3 \quad . \quad . \quad . \quad (4)$$

Multiply (3) by 4:
$$\frac{56}{x} + \frac{48}{z} = 20 \quad . \quad . \quad . \quad (5)$$

Add (4) and (5):
$$\frac{56}{x} + \frac{54}{y} = 23 \quad . \quad . \quad . \quad (6)$$

Now eliminate x from (1) and (6).

Multiply (1) by 28: $\frac{168}{x} + \frac{420}{y} = 112$ . . . (7)

Multiply (6) by 3: $\frac{168}{x} + \frac{162}{y} = 69$ . . . (8)

Subtract (8) from (7): $\frac{258}{y} = 43.$

Hence $y = \frac{258}{43} = 6.$

Substitute $y = 6$ in (1): $\frac{6}{x} + \frac{15}{6} = 4,$

from which $\frac{6}{x} = \frac{3}{2},$

and $x = 4.$

Substitute $y = 6$ in (2): $3 - \frac{16}{z} = 1,$

from which $\frac{16}{z} = 2,$

and $z = 8.$

Answer: $x = 4, y = 6, z = 8.$

These answers are easily verified by substitution in th original equations.

EXERCISES

1. $x - y + z = 3$ . . . (1)
$x + y - z = 3$ . . . (2)
$2x + y - 2z = 1$ . . . (3)

How can y and z be eliminated in one operation? What is th value of x?, of $y - z$? How can x and z be eliminated in on

peration? What is the value of y?, of $x - z$? How is x eliminated om (1) and (2)? What is the result?

2. $$\begin{aligned} x + y &= 15 \quad . \quad . \quad . \quad (1)\\ y + z &= 20 \quad . \quad . \quad . \quad (2)\\ x + z &= 17 \quad . \quad . \quad . \quad (3)\end{aligned}$$

liminate y from (1) and (2). What is the value of $x - z$?
liminate z from (2) and (3). What is the value of $y - x$?
liminate x from (1) and (3). What is the value of $y - z$?

3. $$\begin{aligned} x - 3y + z &= 2 \quad . \quad . \quad . \quad (1)\\ 2x - y - z &= 6 \quad . \quad . \quad . \quad (2)\\ 2x + y - 2z &= 8 \quad . \quad . \quad . \quad (3)\end{aligned}$$

liminate z from (1) and (2). Eliminate y from (2) and (3). liminate x from (2) and (3). Solve the system.

lve the following systems of equations:

4. $$\begin{aligned} 5x - 4y &= 6\\ 6y - 5x &= 1\\ 3z - 2x &= 1.\end{aligned}$$

5. $$\begin{aligned} 2x - 3y + z &= 2\\ x + y - 2z &= 1\\ 3x + 2y - 3z &= 5.\end{aligned}$$

6. $$\begin{aligned} 3x + 2y - 4z &= -2\\ 6x - 4y + z &= 10\\ 2x - 6y + 5z &= 8.\end{aligned}$$

7. $$\begin{aligned} 2x + 4y + 5z &= 49\\ 3x + 5y + 6z &= 64\\ 4x + 3y + 4z &= 55.\end{aligned}$$

8. $$\begin{aligned} 5x + 8y &= 5\\ 3x + 7y + 2z &= 8\\ 2x - 3y - 4z &= 1.\end{aligned}$$

9. $$\begin{aligned} x - y - 2z &= 3\\ 2x + y - 3z &= 11\\ 3x - 2y + z &= 4.\end{aligned}$$

10. $$\begin{aligned} 9x &= 6y = 4z\\ x + y + z &= 38.\end{aligned}$$

11. $$\begin{aligned} x + y &= a\\ y + z &= b\\ z + x &= c.\end{aligned}$$

12. $$\begin{aligned} 5x - 3y &= 8\\ 4y - 3z &= 3\\ 6x - 3z &= y.\end{aligned}$$

13. $$\begin{aligned} \frac{5}{x} - \frac{3}{y} + \frac{1}{z} &= 5\\ \frac{2}{x} + \frac{6}{y} - \frac{5}{z} &= 2\\ \frac{6}{x} + \frac{4}{y} - \frac{3}{z} &= 12.\end{aligned}$$

14. $\frac{6}{x} - \frac{3}{y} + \frac{8}{z} = -6$

$\frac{2}{x} + \frac{9}{y} - \frac{4}{z} = 3$

$\frac{4}{x} + \frac{15}{y} + \frac{12}{z} = 0.$

15. $\frac{10}{x} - \frac{9}{y} = \frac{1}{12}$

$\frac{12}{y} - \frac{9}{z} = \frac{2}{15}$

$\frac{5}{x} - \frac{3}{z} = \frac{1}{10}.$

A Return to Graphs

Some facts regarding graphs were studied in Chapters II a X as well as in Book I. Some of these facts will now be recall and extended for use in solving and interpreting equations.

Linear equations. If an equation in x and y is of the fi degree, its graph is a straight line. It is for this reason that u use the term ***linear equation.***

Quadratic equations. We shall consider here two speci kinds of quadratic equations.

(*a*) The graph of $x^2 + y^2 = 25$ was found to be a circle wi centre at the origin and radius 5. Similarly the graph $x^2 + y^2 = r^2$ is a circle with centre at the origin and radius r.

(*b*) The graph $y = x^2 - 2x - 3$ was plotted on page 222 It is called a ***parabola.*** Other equations whose graphs are par bolas are given in the following exercises (cf. Chapter X.) Y can get a good idea of the shape of a parabola by watchi the path described by a ball when it is thrown into the a Depending on the angle of elevation at which the ball is throw the curve may have a sharp peak or it may be very flat.

EXERCISES

Without plotting them, find the distance from the origin which the graphs of the following equations cross (*a*) the x-ax (*b*) the y-axis:

1. $3x - y = 9$. 2. $x + 6y = 18$. 3. $2x + 5y = 20$.
4. $4x - 3y = 36$. 5. $8x + 9y = 0$. 6. $7x + 2y = 21$.

ive the radius of the circle represented by each of the following:

7. $x^2 + y^2 = 36$. 8. $x^2 + y^2 = 64$. 9. $x^2 + y^2 = 4/9$.
10. $25x^2 + 25y^2 = 16$. 11. $100x^2 + 100y^2 = 81$.
12. $4(x^2 + y^2) = m^2$.

rite the equation of the circle with centre at the origin and th radius as follows:

13. $r = 5$. 14. $r = 7/2$. 15. $r = 2k$.
16. $r = \frac{3}{4}b$.

nstruct the graphs of the following equations:

17. $y = 2x^2$. 18. $y = x^2 + 3x$. 19. $y = 4 - x^2$.
20. $y = x^2 - 4x - 5$.

Other Graphs

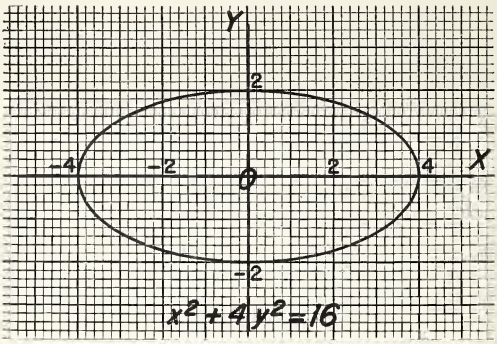
Quadratic equations in x and y may have other kinds of aphs than those considered above. Three other types are ustrated by the following examples.

ample 1. $x^2 + 4y^2 = 16$

If the coefficient 4 were absent from this equation, its graph would be a circle. In what way does the presence of the 4 make the graph differ from a circle? * * * It seems likely, for example, that the graph of $x^2 + 1.1y^2 = 16$ departs very little from the circular shape and that for the given equation the distortion from a circle would be greater. We shall now build up a table of values and see whether this conjecture is verified.

x	-4	-3	-2	-1	0	1	2	3	4
y	0	± 1.32	± 1.73	± 1.94	± 2	± 1.94	± 1.73	± 1.32	0

It is clear that y can have no value if $x > 4$ or $< -$ also x can have no value if $y > 2$ or < -2. When the poin are plotted, the graph is seen to be a symmetrical ova shaped curve. It is called an ***ellipse.***

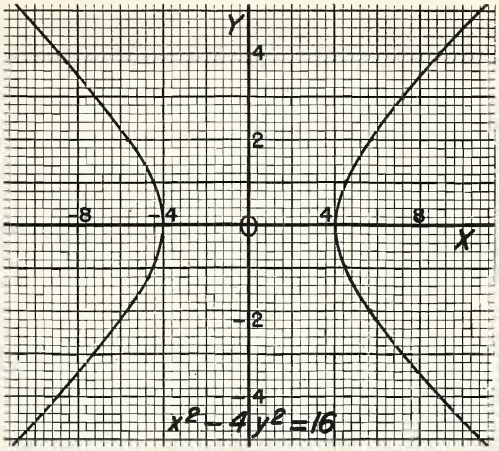


Example 2. $x^2 - 4y^2 = 16$

On account of the minus sign before the second term, t graph of this equation differs in a striking manner from t previous one. To build a table of values we shall assi values to y and calculate corresponding values of x.

y	—4	—3	—2	—1	0	1	2	3	4
x	±8.94	±7.21	± 5.66	±4.47	±4	± 4.47	± 5.66	±7.21	± 8.94

It is clear that x can have no value between -4 and 4 as su a value would make y^2 negative. However, there is no su restriction on the value of y, which can be as large as we lik When the points are plotted, the graph is seen to consist *two branches* symmetrical about the y-axis. This tw branched curve is an example of a ***hyperbola.***



ample 3. $xy = 12$

If the product of two variables remains equal to 12, some conclusions can immediately be drawn. These are:

(1) The two variables have the same sign;

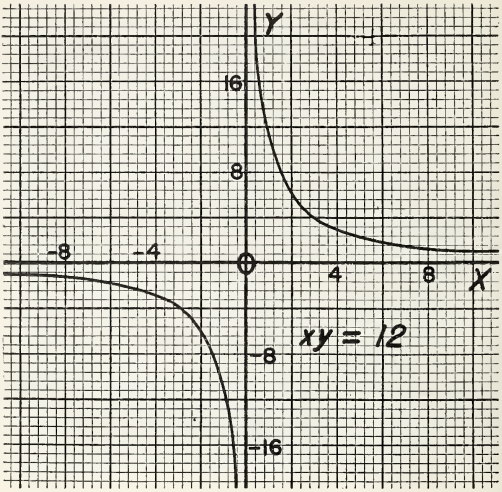
(2) If x is small numerically, y is large numerically, and if x is large y is small;

(3) Neither x nor y can be 0 for, if it were, the product would be 0.

A table of values is easily formed:

x	. . .	1/4	1/3	1/2	1	2	3	4	5	6	7	8	. . .
y	. . .	48	36	24	12	6	4	3	2.4	2	1.7	1.5	. . .

Since both variables can be negative, another table can be formed by writing $-$ before each entry of the preceding table.



What graph does the curve resemble? * * * It is anoth hyperbola but it is placed differently from the former one.

Example 4. $4x^2 - 9y^2 = 0$

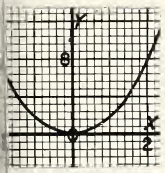
What distinguishes this equation most sharply from ar of the preceding? * * * You observe that (*a*) the rig side is 0 and (*b*) the left side is factorable. The equation c be written

$$(2x - 3y)(2x + 3y) = 0.$$

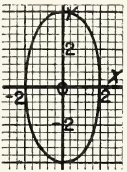
This equation is satisfied when $2x - 3y = 0$ and also wh $2x + 3y = 0$; in other words, it is satisfied by all the poin which lie on either of the two straight lines which are repr sented by these linear equations. This means that the gra of the given equation consists of two straight lines.

The examples which have now been given of graphs quadratic equations embrace all the types which can arise. T

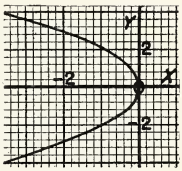
oof of this statement belongs to a more advanced course in athematics. Some equations and their graphs are shown in e following figures.



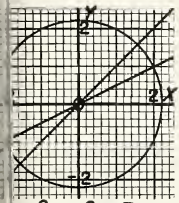
$y = 2x^2$



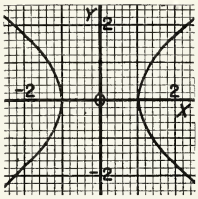
$4x^2+y^2=16$



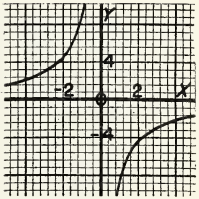
$y^2+4x=0$



$x^2+y^2=5$
$y)(x-2y)=0$



$x^2-y^2=1$



$xy+9=0$

EXERCISES

constructing tables of values, draw the graphs of the following ations:

1. $2x^2 + y^2 = 8$. 2. $x^2 - y^2 = 1$. 3. $xy = 48$.

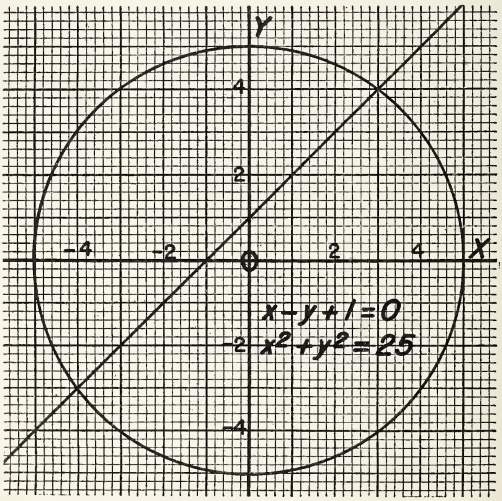
m the following equations pick out those which represent circles, (*b*) ellipses, (*c*) hyperbolas, (*d*) two straight lines.

4. $x^2 - y^2 = 0$. 5. $x^2 + 6y^2 = 20$. 6. $3xy = 25$.
7. $2(x^2 + y^2) = 9$.
8. $(x + y - 1)(2x - y + 3) = 0$.
9. $3x^2 - 5y^2 - 15 = 0$. 10. $25x^2 + 64y^2 = 1600$.
11. $x^2 = 30 - y^2$. 12. $3y^2 = 9 + x^2$.

Systems of One Linear and One Quadratic Equation

Example 1. Solve the system: $x^2 + y^2 = 25$ . . . (1)
$x - y + 1 = 0$. . . . (2)

Graphical solution. These equations are both satisfied by any point which lies on the circular graph of $x^2 + y^2 =$ and on the linear graph of $x - y + 1 = 0$. When the graphs are drawn, they are found to intersect in the points (3, 4) and $(-4, -3)$. The two equations therefore have the solutions $x = 3$, $y = 4$ and $x = -4$, $y = -3$.



Notice that it is not sufficient to say that the equations have the solution $x = \pm 3$, $y = \pm 4$. The pair of values $x = 3$, $y = -4$ is not a solution. It must be specified which value of y is to be taken with each value of x.

Algebraic solution. Let us now consider these two equations from a purely algebraic point of view. From (2) we get

$$y = x + 1 \quad . \quad . \quad . \quad (3)$$

Substituting this value of y in (1), we get

$$x^2 + (x + 1)^2 = 25,$$

or $$2x^2 + 2x + 1 = 25,$$

from which $$x^2 + x - 12 = 0.$$

Hence $$(x - 3)(x + 4) = 0$$

and $$x = 3 \text{ or } -4.$$

If $x = 3$, then, from (3), $y = 4$. If $x = -4$, then $y = -3$.

It follows that the two solutions are $x = 3$, $y = 4$ and $x = -4$, $y = -3$.

The graphical and the algebraic solution of the foregoing set equations illustrate the fusion of algebraic and geometric ideas ich resulted chiefly from the work of the philosopher-thematician, Descartes (1596–1650). Of the two subjects ebra and geometry, each reinforces and illuminates the other.

ample 2. Solve the system: $x^2 + y^2 = 10 \quad . \quad . \quad . \quad (1)$

$$x + 3y = 10 \quad . \quad . \quad . \quad (2)$$

We shall first solve this system algebraically, using ***the method of substitution.*** It is convenient to find x from (2) and substitute in (1). We have

$$x = 10 - 3y,$$

and hence $$(10 - 3y)^2 + y^2 = 10.$$

Thus $$10y^2 - 60y + 90 = 0,$$

from which $$y^2 - 6y + 9 = 0.$$

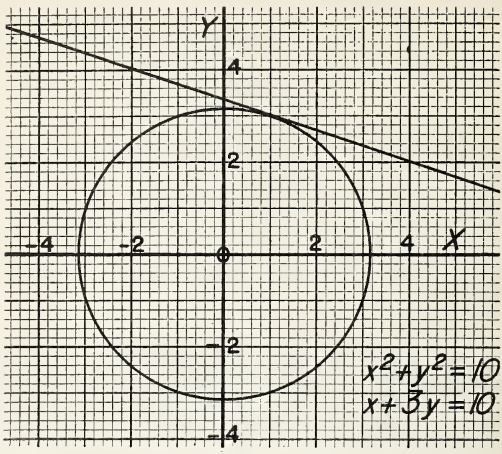
This equation is $$(y - 3)^2 = 0,$$

whence $y = 3$, a double root.

Since $y = 3$, then, from (2), $x = 1$.

Hence the system has only one solution: $x = 1$, $y = 3$.

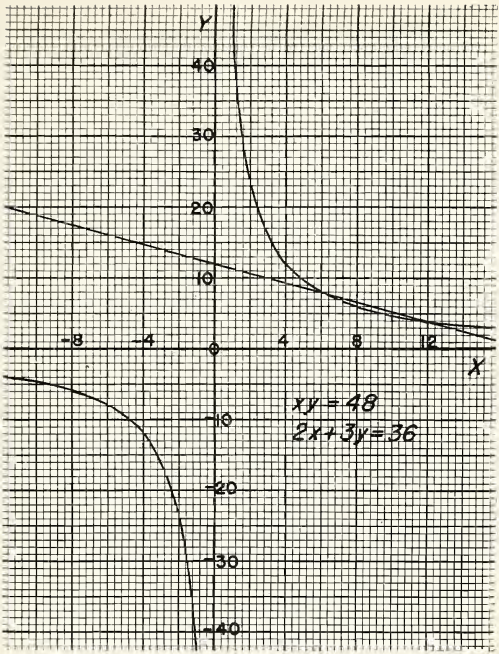
How do the graphs of the circle and straight line show that system has one and only one solution? * * * When the phs are drawn, you see that the line, instead of meeting the



circle in two points, meets it in only one point. In other wor the line is tangent to the circle, as the figure shows. This exam furnishes another illustration of the way in which geometry a algebra can throw light on each other.

Example 3. Solve the system: $xy = 48$ . . . (1
$2x + 3y = 36$ . . . (2

Graphical solution. The graph of the straight line rep sented by (2) cuts the x-axis at the point where $x = 18$ a the y-axis where $y = 12$. The graph of the hyperbola rep sented by (1) was proposed as an exercise (Ex. 3, page 27 When the two graphs are drawn, they are found to inters at the points (6, 8) and (12, 4). This means that the syst of equations has the two solutions $x = 6$, $y = 8$ and $x =$ $y = 4$.



Algebraic solution. We now use the method of substitution. From (2) we find x in terms of y:

$$x = \frac{36 - 3y}{2}.$$

By substituting in (1), we find

$$(36 - 3y)y = 2 \times 48,$$

or $$(12 - y)y = 2 \times 16 = 32.$$

Thus $$y^2 - 12y + 32 = 0,$$

or $$(y - 4)(y - 8) = 0.$$

Hence $y = 4$ or 8.

If $y = 4$, then $x = 12$ and if $y = 8$, $x = 6$; these are t same two solutions as before. Each of these solutions easily shown to satisfy both equations.

Solution of a Linear-quadratic System

The method of substitution: *Solve the linear equation for one of the unknowns in terms of the other, and substitute the result for this unknown in the quadratic.*

Solution by inspection. For some sets of equations of t types just considered, the solutions are easily seen by inspecti For example, to solve

$$x - y = 1$$
$$xy = 12,$$

we require two numbers whose difference is 1 and product It is obvious that $x = 4$ and $y = 3$ are two such numbers. remember, however, that such a set of equations has genera two solutions and a second inspection of the equation shows t $x = -3$, $y = -4$ also satisfy the system.

EXERCISES

Give the solutions of the following systems by inspection:

1. $x + y = 3$, $xy = 2$.
2. $x + y = 11$, $xy = 10$.
3. $x - y = 1$, $xy = 72$.
4. $x - y = 2$, $xy = 15$.
5. $x + 2y = 8$, $xy = 6$.
6. $x^2 + y^2 = 72$, $x - y = 0$.
7. $x^2 + y^2 = 2$, $x + y = 2$.
8. $x^2 + y^2 = 25$, $x + y = 7$.
9. $x^2 + y^2 = 5$, $x + y = 3$.
10. $x^2 + y^2 = 101$, $x + y = 11$.
11. $x^2 + y^2 = 29$, $x - y = 3$.
12. $x^2 + y^2 = 4$, $x - y = 2$.

lve the following systems graphically:

13. $x^2 + y^2 = 25$
$2x+3y = 17.$

14. $x^2 + y^2 = 169$
$2x-3y = 9.$

15. $x^2 + y^2 = 289$
$x + y = 23.$

16. $y - 4x^2 = 0$
$4x - y + 8 = 0.$

17. $2y + x^2 = 0$
$x + y + 4 = 0.$

lve the following systems by the method of substitution:

18. $x^2 + y^2 = 61$
$x - y = 1.$

19. $x^2 + y^2 = 58$
$x + y = 10.$

20. $x^2 + y^2 = 29$
$2x - y = 8.$

21. $x^2 + y^2 = 40$
$3x + y = 12.$

22. $xy = 48$
$x + y = 16.$

23. $xy = 60$
$4x - y = 8.$

24. $xy = 50$
$2x + 3y = 40.$

25. $xy = 36$
$9x - 2y = 3.$

26. $x^2 + y^2 = 146$
$2x - 3y = 7.$

27. Solve the problem on page 260: Find the sides of a rectangle whose perimeter is 42 feet and area 108 square feet.

urther Examples Involving Quadratic Equations

ample 1. Solve:

$$x^2 + y^2 = 97 \quad . \quad . \quad . \quad (1)$$
$$xy = 36 \quad . \quad . \quad . \quad (2)$$

One method of solving this set is by substituting $36/x$ for y in (1). Another method is the following:

Since
$$2xy = 72, \quad . \quad . \quad . \quad (3)$$
we get, by adding (3) to (1),
$$(x + y)^2 = 97 + 72 = 169, \quad . \quad . \quad . \quad (4)$$
and, by subtracting (3) from (1),
$$(x - y)^2 = 97 - 72 = 25 \quad . \quad . \quad . \quad (5)$$

Hence, by taking the square roots of (4) and (5), we have

$$x + y = \pm 13$$
$$x - y = \pm 5.$$

This gives the four sets of equations:

$x + y = 13$ $x + y = 13$ $x + y = -13$ $x + y = -1$
$x - y = 5$ $x - y = -5$ $x - y = 5$ $x - y = -$

These equations yield the solutions:

$x = 9$ $x = 4$ $x = -4$ $x = -9$
$y = 4$ $y = 9$ $y = -9$ $y = -4$

These four answers are easily verified.

Example 2. Solve: $x^2 - 3xy + 2y^2 = 0$ . . . (1)
$x^2 + y^2 = 5$ . . . (2)

Notice that in (1) the right member is 0 and the terms o the left are all of degree 2. This is expressed by saying th the equation is ***homogeneous of the second degree.*** Is possible to obtain from (1) two equations of the first degre * * * It is clear that this can be done and the clue to t method is to factor the left side of (1). This gives

$(x - y)(x - 2y) = 0$ . . . (3)

Equation (3) is satisfied by $x - y = 0$ and also by $x - 2y =$ Hence the complete solution is obtained by solving the tw systems

$x - y = 0$ and $x - 2y = 0$
$x^2 + y^2 = 5$ $x^2 + y^2 = 5.$

These solutions are carried out by the method of substitutio The details are left as an exercise.

Graphical meaning of these solutions. Since (1) is equivale to the two equations $x - y = 0$ and $x - 2y = 0$, its graph co sists of two straight lines. Each of these lines cuts in two poin the circle represented by (2) (see figure, page 273). This accoun for the four solutions of the given system.

Example 3. Solve: $2x^2 - 2xy - 5y^2 = 11$ . . . (1)
$x^2 - 3xy + 2y^2 = 4$ . . . (2)

Can we obtain from these a linear equation from which might substitute? In Example 2 the first equation w

homogeneous, which suggested the device of factoring. In these equations, although the left members are homogeneous, the right members are not 0. From these equations how can we obtain an equation in which the left side is homogeneous and the right side 0? * * * It is clear that we should ***eliminate the constant terms.*** This is done by multiplying (1) by 4, (2) by 11, and subtracting. This gives

$$8x^2 - 8xy - 20y^2 = 44$$
$$11x^2 - 33xy + 22y^2 = 44.$$

Hence $3x^2 - 25xy + 42y^2 = 0.$

This gives, by factoring, $(x - 6y)(3x - 7y) = 0$

from which $x = 6y$ or $7y/3.$

The solution is most easily completed by substituting in (2). The details are as follows:

Substitute $x = 6y$ in (2).	Substitute $x = 7y/3$ in (2).
Then $(36 - 18 + 2)y^2 = 4,$	Then $(\frac{49}{9} - 7 + 2)y^2 = 4,$
or $20y^2 = 4,$	or $\frac{4}{9}y^2 = 4,$
whence $y^2 = \frac{1}{5}$	whence $y^2 = 9$
and $y = \pm \frac{1}{\sqrt{5}} = \pm \frac{\sqrt{5}}{5}.$	and $y = \pm 3.$
Then $x = \pm \frac{6\sqrt{5}}{5}.$	Then $x = \pm 7.$

The four solutions are:

$$x = 7, \quad x = -7, \quad x = \frac{6\sqrt{5}}{5}, \quad x = -\frac{6\sqrt{5}}{5},$$

$$y = 3, \quad y = -3, \quad y = \frac{\sqrt{5}}{5}, \quad y = -\frac{\sqrt{5}}{5}.$$

EXERCISES

Solve:

1. $x^2 + y^2 = 100$
   $xy = 48.$

2. $4x^2 - y^2 = 0$
   $10x^2 + 3xy = 16.$

3. $x^2 + y^2 + 4x - 5y = 45$
   $x^2 - y^2 = 0.$

4. $6x^2 - xy - y^2 = 0$
   $x^2 + y^2 = 20.$

5. $3x^2 - 5y^2 = 63$
   $xy = 18.$

6. $2x^2 - 11y^2 = 28$
   $xy + 4y^2 = 4.$

7. $x^2 + 2xy + 5 = 0$
   $xy + 3y^2 = 12.$

8. $x^2 - y^2 = 17$
   $xy = 72.$

9. $x^2 + xy = 28$
   $xy - y^2 = -30.$

10. $x^2 - xy - 2y^2 = 0$
    $3x^2 - 2y^2 = 40.$

11. Show, without plotting them, that the graphs of $x^2 + y^2 = 10$ and $xy = 20$ do not intersect.

Solve:

12. $y = x^2 - 4x$
    $x^2 - 4y^2 = 0.$

Sketch a rough graph to show t
interpretation of your solution.

13. $17x^2 - 20xy - 14y^2 - 169 = 0$
    $7x^2 - 22xy - 31y^2 + 169 = 0.$

14. $2x^2 + xy - 15y^2 = 0$
    $x^2 - xy - 2y^2 = 28.$

15. $x^2 - 3xy + 2y^2 = -4$
    $3x^2 + xy - 5y^2 = 13.$

Application to Problems

It often happens that a problem can be solved by introduci
one unknown or by introducing two unknowns. The example
the digits of a number, solved on page 167, can also be solved
using two letters for the two unknown digits.

The following examples illustrate the use of two or of thr
unknowns.

Example 1. If the numerator and denominator of a fraction a
each increased by 4, the fraction can be reduced to 7/9; if t
terms are each decreased by 3, it can be reduced to 3/4. Fi
the fraction.

Denote the fraction by $\frac{x}{y}$. The conditions of the problem then give

$$\frac{x+4}{y+4} = \frac{7}{9} \quad . \quad . \quad (1)$$

and

$$\frac{x-3}{y-3} = \frac{3}{4} \quad . \quad . \quad . \quad (2)$$

From (1) $9x + 36 = 7y + 28,$
and from (2) $4x - 12 = 3y - 9.$
These reduce to $9x - 7y = -8,$
$4x - 3y = 3.$

These equations yield the solution $x = 45$, $y = 59$.
Hence, the required fraction is $\frac{45}{59}$.

Verification: $\frac{45+4}{59+4} = \frac{49}{63} = \frac{7}{9}$, and $\frac{45-3}{59-3} = \frac{42}{56} = \frac{3}{4}$.

xample 2. A mixture is made of three substances A, B, and C. ne quart of A weighs 20 oz. and costs 75c. One quart of B eighs 15 oz. and costs 65c. One quart of C weighs 24 oz. and sts 50c. The total mixture of 18 quarts weighs $23\frac{1}{4}$ lb. and sts $11.10. Find the amount of each substance used.

Let the mixture contain x quarts of A, y quarts of B, and z quarts of C. We get three equations by expressing the total number of quarts, the total number of ounces and the total number of cents. These equations are

$$x + y + z = 18 \quad . \quad . \quad . \quad (1)$$
$$20x + 15y + 24z = 23\tfrac{1}{4} \times 16 = 372 \quad . \quad . \quad . \quad (2)$$
$$75x + 65y + 50z = 1110 \quad . \quad . \quad . \quad (3)$$

Elimination of z between (1) and (2) gives

$$4x + 9y = 24 \times 18 - 372 = 60 \quad . \quad . \quad . \quad (4)$$

Elimination of z between (1) and (3) gives

$$25x + 15y = 210$$

or $5x + 3y = 42.$ . . . (5

Elimination of y between (4) and (5) gives

$$11x = 66.$$

Hence $x = 6.$

Then from (5) $30 + 3y = 42,$

and $y = 4.$

Now from (1) $6 + 4 + z = 18,$

and $z = 8.$

Hence the mixture contains 6 quarts of A, 4 of B, and 8 of C

Example 3. The perimeter of a rectangular field measures 68 rod and its diagonal 26 rods. Find the dimensions of the field.

Let the sides of the field be x rods and y rods, respectively. Then the perimeter is $2x + 2y$ rods and the diagonal is $\sqrt{x^2 + y^2}$ rods.

Hence $2x + 2y = 68$

and $\sqrt{x^2 + y^2} = 26.$

These give, by simplification, $x + y = 34$

and $x^2 + y^2 = 26^2.$

Substituting for y from the first equation in the second, we get

$$x^2 + (34 - x)^2 = 26^2.$$

Hence $2x^2 - 68x + 34^2 - 26^2 = 0,$

or $2x^2 - 68x + (34 + 26)(34 - 26) = 0.$

That is $2x^2 - 68x + 60 \times 8 = 0$

or $x^2 - 34x + 240 = 0,$

or $(x - 24)(x - 10) = 0.$

Hence, $x = 24$ or $10.$

If $x = 24$, $y = 10$, and if $x = 10$, $y = 24$. Hence the dimensions of the field are 10 rods and 24 rods.

EXERCISES

1. If the numerator of a fraction is increased by 1, the fraction can be reduced to $\frac{4}{7}$; if the denominator is decreased by 1, it can be reduced to $\frac{5}{9}$. Find the fraction.

2. If the numerator of a fraction is increased by 3 and the denominator decreased by 3 the new fraction is equal to 1; if the numerator is decreased by 3 and the denominator increased by 3, the new fraction is equal to $\frac{1}{4}$. Find the fraction.

3. The sum of the digits in a two-digit number is 13. If the digits are reversed, the number is increased by 27. Find the number.

4. A 2-digit number is equal to 6 times its units digit. The sum of the number and the number obtained by reversing the digits is 132. Find the number.

5. Two numbers consist of the same two digits in opposite orders. The sum of the numbers is 110 and the difference is 18. Find the numbers.

6. The sums of three numbers taken in pairs are 17, 22, and 25. Find the numbers.

7. Of three numbers, whose sum is 72, the second is the average of the first and third, and the third is the average of the second and twice the first. Find the numbers.

8. A father is twice as old as his son. Twelve years ago he was three times as old as his son. Find their ages.

9. The ages of a father, his son, and his grandson total 123 years. At the time when the grandson was born, the ages of the father and son totalled 75 years. Nine years from now the father's age will be the sum of those of his son and grandson. Find their ages.

10. A man has \$1200 invested in bonds, some of which bear 3% coupons and the rest $3\frac{1}{2}\%$ coupons. If his yearly income from the bonds is \$37, find the amount invested at each rate.

11. A man's savings of \$2300 are invested, part at 3%, part at 4%, and the remainder at 2%, per annum. His annual interest is \$66 and his interest on the 3% investment is the same as on the 4% investment. Find the amount invested at each rate.

12. Before the chaff was cleaned from some wheat by a fanni mill, a bushel of the mixture weighed 55 pounds. If whe weighs 60 pounds to the bushel and chaff 10 pounds, find t weight of wheat in a bushel of the mixture.

13. A farmer mixed 10 bushels of feed, using the following i gredients: (1) bran, which weighs 20 pounds to the bushel a costs 40c. a bushel; (2) crushed oats, which weigh 34 pounds the bushel and cost 65c. a bushel; (3) ground corn, which weig 48 pounds to the bushel and costs 80c. a bushel. The tot mixture weighed 326 pounds and cost $6.05. Find the amou of each grain in the mixture.

14. In an election contested by three candidates, the su cessful candidate received 3/7 of the total number of votes ca He received 1080 votes more than his nearest opponent, w received 1800 more than the third candidate. How many vot did each candidate get?

15. Three samples of metals, A, B, and C, are weighed in balance. When A and B are in one scale pan and C in the oth it takes 35 grams along with C to produce balance. When A a C are in one scale pan and B in the other, it takes 47 grams wi B to produce balance. When B and C are in one pan and A the other, it takes 3 grams with B and C to produce balanc Find the weights of A, B, and C.

16. Of two numbers, the first exceeds the second by 9 and exceeds twice the square of the second by 3. Find the numbe

17. Of two numbers, the first exceeds the second by 8 a falls short of the square of the second by 34. Find the numbe

18. The sum of the squares of two numbers exceeds the su of the numbers by 92. The difference of the squares exceeds t difference of the numbers by 52. Find the numbers.

19. The perimeter of a rectangle is 62 cm. and the diagon is 25 cm. Find the sides.

20. An open rectangular box has a square base. The outsi area is 176 sq. in. The sum of the three edges which meet at a corner is 18 in. Find the dimensions.

EXERCISES ON CHAPTER XII

A

olve the following systems of equations:

1. $3x + 5y = 171$
   $7x + z = 209$
   $7y + z = 90.$

2. $x - 3y + z = 2$
   $2x + y - 2z = 3$
   $4x - 2y - 3z = 4.$

3. $2x - 3y + 4z = 8$
   $3x + y - 7z = 14$
   $x - 2y + 3z = -2.$

4. $7x - 3y = 30$
   $9y - 5z = 34$
   $x + y - z = 7.$

5. $\frac{1}{x} + \frac{2}{y} = \frac{7}{6}$
   $\frac{2}{x} - \frac{6}{y} = -1.$

6. $\frac{4}{x} + \frac{6}{y} + 1 = 0$
   $\frac{1}{x} - \frac{2}{y} = 5.$

7. $\frac{8}{x} + \frac{9}{y} = 43$
   $\frac{9}{x} + \frac{8}{y} = 42.$

8. $x - y = 0$
   $x^2 + 3xy + y^2 = 1.$

9. $x + y = 12$
   $xy = 35.$

10. $x^2 + y^2 = 25$
    $x + y = 7.$

11. $x^2 - y^2 = 28$
    $x + y = 14.$

12. $x^2 - y^2 = 80$
    $x - y = 4.$

13. $x^2 + y^2 = 29$
    $xy = 10.$

14. $x^2 + y^2 = 40$
    $x^2 - y^2 = 32.$

15. A fraction can be reduced to $\frac{1}{2}$ when its denominator is creased by 2 and can be reduced to $\frac{1}{4}$ when its numerator is creased by 1. Find the fraction.

16. If the terms of a fraction are each diminished by 1, the action can be reduced to 4/7. If the terms are each increased 2, it can be reduced to 3/5. Find the fraction.

17. Two numbers consist of the same two digits in opposite ders. The sum of the numbers is 143 and their difference is 27. nd the numbers.

18. The sum of the three digits of a number is 15. If the firs two digits are interchanged, the number is increased by 27(If the first and third digits are interchanged, the number is in creased by 594. Find the number.

19. A's and B's ages are in the ratio 3 : 4. Twelve year ago they were in the ratio 5 : 8. Find their ages.

20. The sum of the ages of a man, his son, and his grandso is 96 years. When the grandson was born the man was 6 year more than twice as old as his son. Twenty years from now th man's age will be the sum of those of his son and grandson. Fin their ages.

21. The perimeter of a rectangle is 78 in. and the area i 360 sq. in. Find the sides.

22. The sum of the areas of two squares is 106 sq. in. and th sum of their perimeters is 56 in. Find their sides.

B

Solve:

23. $\frac{4}{x}+\frac{3}{y}=\frac{12}{y}-\frac{4}{z}=\frac{2}{x}+\frac{3}{z}=1.$

24.
$$\tfrac{1}{3}(x-y)+z=2$$
$$\tfrac{1}{2}(x+z)+y=5$$
$$\tfrac{1}{2}(x-z)-\tfrac{1}{3}(y+z)=1.$$

25.
$$x+\tfrac{1}{2}y+\tfrac{1}{3}z=32$$
$$\tfrac{1}{3}x+\tfrac{1}{4}y+\tfrac{1}{5}z=15$$
$$\tfrac{1}{4}x+\tfrac{1}{5}y+\tfrac{1}{6}z=12.$$

26.
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=10$$
$$\frac{2}{x}-\frac{1}{y}+\frac{1}{z}=2$$
$$\frac{3}{x}+\frac{2}{y}-\frac{3}{z}=7.$$

27.
$$xy=3(x+y)$$
$$xz=8(x+z)$$
$$7yz=24(y+z).$$

28.
$$3y-5x=xy$$
$$2y+3x=26xy.$$

29. $x^2 + y^2 = 289$
$2x + y = 38.$

30. $x^2 + 5xy + y^2 + 17 = 0$
$x + y = 5.$

31. $x^2 + y^2 = 5$
$y^2 - xy = 3.$

32. $3x^2 - 5y^2 = 28$
$3xy - 4y^2 = 8.$

33. $4x^2 + 9xy + 9y^2 = 11$
$4x + 6y = 3.$

34. $x^2 - xy + 3y^2 = 9$
$2x^2 + 3xy = 2.$

35. $2x^2 - 3xy + 2y^2 = 14$
$x^2 - xy = 4.$

36. $x^2 + xy + 2y^2 = 44$
$2x^2 - xy + y^2 = 16.$

37. $3x^2 + 165 = 16xy$
$7xy + 3y^2 = 132.$

38. $x^4 + x^2y^2 + y^4 = 21$
$x^2 - xy + y^2 = 3.$

39. The perimeter of a rectangle is 46 in. and the diagonal is 7 in. Find the sides.

40. The sum of the base and the altitude of a triangle is 22 in. nd the area is 56 sq. in. Find the base and the altitude.

41. A rectangular room has a square floor and ceiling. The tal area of the walls, floor, and ceiling is 672 sq. ft. The sum the lengths of the 12 edges of the room is 128 ft. Find the mensions of the room, given that the height is the smallest mension.

42. Two customers spent the same amount in buying tea at)c. a pound and coffee at 60c. a pound. The first bought equal eights of tea and coffee. The second spent the same amount r each. The second bought altogether $\frac{1}{2}$ pound more than the st. How many pounds of each did each customer buy?

43. A truck carries a load of crated furniture of which there e three different kinds of articles. Each of the crates of one nd occupies 12 cu. ft. and weighs 50 lb.; each of the second nd occupies 8 cu. ft. and weighs 60 lb.; each of the third kind cupies 9 cu. ft. and weighs 85 lb. The total load consists of crates occupying 184 cu. ft. and weighing 1280 lb. Find the mber of each kind.

C

44. A balance is furnished with weights of three different kinds, A, B, and C. It will weigh 5 grams if we put one A and one C in one scale pan and two B's together with the 5 grams in the other. It will weigh 11 grams if we put one A and one B in one scale pan and one C together with the 11 grams in the other. It will weigh 15 grams if we put 4 B's and 3 C's in one scale pan and one A together with the 15 grams in the other. Find the weights of A, B, and C.

45. It is required to mix a 4% solution of salt with a 12% solution to obtain 8 pounds of a 10% solution. What weight of each solution should be used?

46. A plays checkers with B, winning 4 games to every 3 for B. A plays with C, winning 3 games to every 2 for C. When A has played a total of 48 games, he has won 28. How many games did he play with each opponent?

47. Three typists, A, B, and C, type a book of 193 pages. A types the first part at 6 pages an hour, then B relieves her and does a part at 8 pages an hour. B is relieved by C, who does 7 pages an hour. The book is completed in 28 hours and C does 11 pages less than A and B combined. How many pages are done by each typist?

48. The sum of the digits of a 3-digit number is 15. If the first two digits are reversed, the number is diminished by 180. If the second and third digits are reversed, the number is diminished by 18. Find the number.

49. To diminish each term of a fraction by unity diminishes the fraction by 1/12; to increase each term by unity increases the fraction by 1/20. Find the fraction.

50. Show that, if a fraction can be reduced to $\frac{1}{2}$ when its denominator is decreased by 2, then it can also be reduced to $\frac{1}{2}$ when its numerator is increased by 1.

51. In a 300-yard race A finishes 20 yards ahead of B and

yards ahead of C. Assuming the speed of each runner to be nstant, find how far B finishes ahead of C.

52. Show that the values of x and y which satisfy the system

$$x + y = a$$
$$xy = b$$

the roots of the equation $r^2 - ar + b = 0$.

53. The formula for the number of feet which a stone falls om rest in t seconds is $s = 16t^2$. Two observations on a falling one showed that it fell 128 feet in a certain 2 consecutive conds. How long had the stone been falling when the first servation was made?

CHAPTER XIII

LOGARITHMS

We shall now open the door to a new method of arithmetic computation, a method which seemed to the people who first used it over 300 years ago like a bit of mathematical magic. And yet its usefulness is matched by its simplicity—it is merely a matter of making judicious use of indices.

Recall the laws of indices as they are used to find:

(*a*) *the product of two powers:* $a^m \times a^n = a^{m+n}$;

(*b*) *the quotient of two powers:* $a^m \div a^n = a^{m-n}$;

(*c*) *a power of a power:* $(a^m)^r = a^{mr}$;

(*d*) *a root of a power:* $\sqrt[p]{a^m} = a^{m/p}$.

EXERCISES

Express each of the following as a power of 10:

1. $10^4 \times 10^5$.
2. $10^3 \times 10^7 \times 10^{-2}$.
3. $10^9 \div 10^6$.
4. $(10^5)^3$.
5. $(10^{3.5})^2$.
6. $(10^{-4})^{-3}$.
7. $\sqrt{10^8}$.
8. $\sqrt[3]{10^{5.31}}$.
9. $\sqrt[5]{10^{-2.5}}$.

A New Use of Indices

You have learned that a^m is a ***power*** of a, in which a is called the ***base*** and m the ***index*** or ***exponent.*** In this chapter we shall use for base the number 10.

The powers of 10. It is easy to write the values of the integral powers of 10:

$10^0 = 1$; $10^3 = 1000$; $10^7 = 10{,}000{,}000$; $10^{-2} = 0.01$; $10^{-5} = 0.00000$

and so on. Also, since $10^{0.5} = 10^{\frac{1}{2}} = \sqrt{10}$, its value can be found

$$10^{0.5} = 3.162, \text{ approximately.}$$

, however, we want the value of $10^{2.6}$ or of $10^{-1.8}$, we cannot id them so easily. Again, if we should desire the power of 10 ich is equal to 37.8, we should find the problem beyond us.

Suppose now that whenever we wish to know the numerical lue of any power of 10, or the power of 10 which is equal to any ven number, we have only to summon a magician who will esent us with the answer. Suppose, for example, we are told at

$$37.8 = 10^{1.577} \text{ and } 2.34 = 10^{0.369};$$

w could we find the product 37.8×2.34?

* * *

ou see that $37.8 \times 2.34 = 10^{1.577 + 0.369}$
$= 10^{1.946}$.

ain we summon the magician who tells us that $10^{1.946} = 88.3$.

ence $37.8 \times 2.34 = 88.3$,

e result being correct to 3 figures. The only actual computa- n we performed was the addition of the two indices, 1.577 + 69.

If you stop to think for a moment, you will see that the thod has great possibilities, not only for multiplying numbers t also for dividing numbers and for finding powers and roots numbers. All we need is to be able to convert numbers into wers of 10 and powers of 10 into numbers! Let us assume t our magician has presented us with a book giving the swers to all such questions.

ample 1. Divide 946.2 by 385.9.

We find, from our book of indices, that

$$946.2 = 10^{2.9760} \text{ and } 385.9 = 10^{2.5865}.$$

Hence the index law tells us that

$$946.2 \div 385.9 = 10^{2.9760-2.5865} = 10^{0.3895},$$

and now the book of indices gives us the required answer:

$$10^{0.3895} = 2.452.$$

Hence, finally, $946.2 \div 385.9 = 2.452$, correct to 4 figure
The only computation we did was the subtraction of two indice

Example 2. Find the value of $\sqrt{8646}$.

We first look up 8646 as a power of 10:

$$8646 = 10^{3.9368}.$$

Then
$$\sqrt{8646} = (10^{3.9368})^{\frac{1}{2}} = 10^{1.9684},$$

which we find to be 92.98, correct to 4 figures.

By this time you will be getting curious about our magic boo
of indices.

Let us first notice the ***approximate character of mo
of these indices.*** It is easy to see that $10000 = 10^4$, but wh
power of 10 is 1248? For such a power of 10, the index is not
whole number; moreover, when the index is written in decim
form, the decimal does not terminate and the best we can do
to approximate to it with some convenient number of figures.

For the present we shall use without question the values
these indices which our predecessors have worked out for us a
recorded in books. The finding of them now would be a difficu
problem for us. If you study mathematics for a few years mo
however, you will find that the indices are, after all, quite easi
calculated.

> ***It is a simple matter to find products, quotients, powers, and roots of numbers, provided we are given any number as a power of 10 and any power of 10 as a number in decimal form.***

A New Name

In order to use our new method of computation, all we need
e:

) *the indices of the powers of 10 which correspond to given numbers,* and
) *the numbers which correspond to given indices.*

ith this in mind, we might write the solution of the division
oblem in *Example* 1 as follows:

Index for 946.2 = 2.9760,
Index for 385.9 = 2.5865,
(Subtract) 0.3895,

which is the index for 2.452.

Therefore 946.2 ÷ 385.9 = 2.452.

When an index of a power of 10 is used for this purpose, it is
lled a ***logarithm.*** This rather formidable word is descended
m two Greek words which may be freely translated as *the*
ence of the number.

> ***When a number is expressed as a power of 10, the index of the power is called the logarithm of the number.***

The equation $1000 = 10^3$
n now be written as $\log 1000 = 3,$
d, in general $N = 10^l$
n be written as $\log N = l.$

EXERCISES

ite the following equations in logarithmic form:

1. $10^2 = 100.$
2. $10^0 = 1.$
3. $10^{-2} = 0.01.$
4. $10^{-5} = 0.00001.$

5. $10^{\frac{1}{2}} = \sqrt{10}$.

6. $10^{\frac{2}{3}} = \sqrt[3]{100}$.

7. $10^{r} = s$.

8. $10^{m/n} = \sqrt[n]{10^{m}}$.

9. $10^{-\frac{3}{2}} = \dfrac{1}{10\sqrt{10}}$.

10. $10^{-1/p} = \dfrac{1}{\sqrt[p]{10}}$.

Write the following equations in a form in which the symbol $\log$ does not appear:

11. $\log 10000 = 4$.

12. $\log 1\,000\,000 = 6$.

13. $\log 0.1 = -1$.

14. $\log 0.0001 = -4$.

15. $\log \sqrt[5]{1000} = 3/5$.

16. $\log \sqrt[q]{10^{p}} = p/q$.

17. $\log 2 = 0.30103$.

18. $\log 30 = 1.47712$.

19. $\log h = k$.

20. $\log 24.56 = 1.3902$.

Write the values of the following:

21. $\log 10$.

22. $\log 100\,000$.

23. $\log 1$.

24. $\log 1/10$.

25. $\log 10^{-6}$.

26. $\log \sqrt[3]{10}$.

27. $\log \sqrt[5]{10}$.

28. $\log \sqrt[4]{1000}$.

29. $\log 1/\sqrt{10^{4}}$.

30. $\log 0.01$.

31. $\log 0.001$.

32. $\log 10^{-2p}$.

A Useful Fact about Logarithms

When the logarithms of numbers have been found, they a written in a ***table*** or, as we say, ***tabulated.*** This is o example of many such tables (including, for example, tables squares and square roots), all of which are for the purpose saving work; tables are labour-savers. As we have already sa we shall not here inquire into the manner in which logarithn tables are made; our task is to learn how to use them.

We now illustrate an important property of logarithms. om the tables we find that

$$\log\ 56.84 = 1.7546,$$

ich means that $56.84 = 10^{1.7546}$.

follows that $568.4 = 10^{1.7546} \times 10 = 10^{2.7546}$,

d that $5684 = 10^{1.7546} \times 10^2 = 10^{3.7546}$.

so $5.684 = 10^{1.7546} \div 10 = 10^{0.7546}$,

d so on. Each logarithm (that is, each index) consists of an egral part and a fractional part. Thus the logarithm 1.7546 s the integral part 1 and the fractional part .7546. From the egoing example the following important fact now appears.

A change in the position of the decimal point in a number gives rise only to a change in the integral part of its logarithm.

This fact requires further study. To continue the foregoing mple, we have

$$0.5684 = 10^{1.7546} \times 10^{-2} = 10^{-1+.7546}$$
$$0.005684 = 10^{1.7546} \times 10^{-4} = 10^{-3+.7546}$$

e logarithm $-1 + .7546$ can be written as -0.2454 but it is ch better to leave it in the first form, in which the fractional t is positive. If we adopt this convention, we have the lowing result.

The fractional part of the logarithm of a number depends only on the sequence of its digits, and not on the position of the decimal point.

Definition. *The integral part of a logarithm is called its* ***aracteristic;*** *the fractional part, always taken positive, is ed its* ***mantissa.***

You see now the advantage of making the mantissa alwa positive; we need only one entry in the table for the logarith of 0.005684, 0.05684, 0.5684, 5.684, 56.84, and so on. T makes for economy in the building of tables.

Position of decimal point	*determines*	***characteristic.***
Significant digits	*determine*	***mantissa.***

The Rule for Finding the Characteristic

The standard form for writing a number. The follow equations are easily verified:

$$23.46 = 2.346 \times 10,$$
$$1873.5 = 1.8735 \times 10^{3},$$
$$0.586 = 5.86 \times 10^{-1},$$
$$0.0075 = 7.5 \times 10^{-3}.$$

The numbers on the right are said to be expressed in ***standa form.*** Now pause and frame your own definition.

* * *

When a number is expressed as the product of a power of 10 times a number with one significant (non-zero) digit before the decimal point, it is said to be expressed in ***standard form.***

EXERCISES

Write the following numbers in standard form:

1. 784.	**2.** 65.32.	**3.** 3160.
4. 240000.	**5.** 93 000 000.	**6.** 346.5.
7. 10500.	**8.** 0.562.	**9.** 0.0703.
10. 50.05.	**11.** 0.00084.	**12.** 0.00589.

13. 57380. 14. 0.504. 15. 0.000095.
16. 37526. 17. 0.000006. 18. 0.005004.

Examine the following equations:

$$10^{-3} = 0.001 \quad \text{or} \quad \log 0.001 = -3.$$
$$10^{-2} = 0.01 \quad \text{or} \quad \log 0.01 = -2.$$
$$10^{-1} = 0.1 \quad \text{or} \quad \log 0.1 = -1.$$
$$10^{0} = 1 \quad \text{or} \quad \log 1 = 0.$$
$$10^{1} = 10 \quad \text{or} \quad \log 10 = 1.$$
$$10^{2} = 100 \quad \text{or} \quad \log 100 = 2.$$
$$10^{3} = 1000 \quad \text{or} \quad \log 1000 = 3.$$

It follows that if any number lies between 0.001 and 0.01 its garithm lies between -3 and -2. Hence its logarithm can e taken as -3 ***plus*** a fraction or as -2 ***minus*** a fraction. In der that the mantissa be positive, we use the former and ke the characteristic as -3. Similarly, if a number is between 01 and 0.1 the characteristic of its logarithm is -2, and so on. ow consider the numbers which were written above in standard rm:

umber	*Located between*	*Standard form*	*Characteristic*
23.46	10 and 10^2	2.346×10	1
873.5	10^3 and 10^4	1.8735×10^3	3
0.586	10^{-1} and 10^0	5.86×10^{-1}	-1
0.0075	10^{-3} and 10^{-2}	7.5×10^{-3}	-3

When you examine the foregoing table a useful fact appears. order to express 1873.5 in standard form, the decimal point moved 3 places to the left and the characteristic of the logarm is 3. In order to express 0.586 in standard form, the cimal point is moved one place to the right and the characteric of the logarithm is -1. Examine the other two numbers the table and make up other examples of your own. You ll soon be convinced of the truth of the following rule.

To find the characteristic of the logarithm of a number:

(1) *mentally change the number into standard form;*

(2) *if this change moves the decimal point n places to the* ***left,*** *the characteristic is* $+n$*; if this change moves the decimal point n places to the* ***right,*** *the characteristic is* $-n$.

Example. Give the characteristics of the logarithms of 825 and 0.00006.

In standard form, the first number would have the decim point between 8 and 2, that is 4 places to the left of i present position. Hence the characteristic of its logarith is $+4$.

The second number, in standard form, would have the decim point after the 6, 5 places to the right of its present positio Hence the characteristic of its logarithm is -5.

EXERCISES

Give the characteristics of the logarithms of the followi numbers:

1. 7.	**2.** 83.	**3.** 560.
4. 26.4.	**5.** 78.23.	**6.** 685.9.
7. 1.356.	**8.** 0.482.	**9.** 0.066.
10. 7426.	**11.** 37000.	**12.** 1728.25.
13. 580.7.	**14.** 9.9.	**15.** 10.01.
16. 0.5002.	**17.** 0.00625.	**18.** 0.00072.
19. 3.008.	**20.** 700.28.	**21.** 6.666.
22. 0.0125.	**23.** 0.00008.	**24.** 0.002002.

The Use of Tables

We have seen that it is sufficient to tabulate the mantissas f logarithms; the characteristics can be read by inspection of ıe numbers.

How to find the mantissa. On pages 320 and 321 is given table of mantissas of logarithms. Two features of this table -e called to your attention:

(1) *The decimal points are omitted. When you use a mantissa om the table, you must insert a decimal point at the left of the first git.*

(2) *Each mantissa contains four digits.* This does not mean ıat the mantissa is given exactly by these four digits but only)proximately, *correct to four figures.* Other tables are often ;ed with more figures in the mantissas. These are spoken of 5-figure tables, 7-figure tables, and so on, according to the ımber of figures given. The number of figures used by en- neers and other computers depends on the degree of accuracy their data and of their desired results.

If a number has one, two, or three digits, its mantissa can read directly from the table, as described in the following ample.

xample 1. Find (*a*) log 2, (*b*) log 68, (*c*) log 43.7.

(*a*) log 2 is found by locating 20 in the left-hand column of the table and reading from the first column of mantissas, headed 0. The entry is 3010. Since the characteristic is 0, we have

$$\log 2 = 0.3010.$$

(*b*) In a similar manner, the entry for 68 is found to be 8325. Affixing the characteristic, which is 1, we have

$$\log 68 = 1.8325.$$

(*c*) For log 43.7 we locate 43 in the left-hand column of numbers and 7 in the top horizontal column. The corres-

ponding entry in the table is 6405. Affixing the characteristic 1, we have

$$\log 43.7 = 1.6405.$$

If a number has four digits, its mantissa does not occur in the table but is intermediate between two adjacent mantissas of the table. The correct value is found by adding to the smaller one of these mantissas or subtracting from the larger one. To facilitate this work, a table of differences is given on the right of the table of mantissas. The manner in which this is used is indicated in the following examples.

Example 2. Find log 57.24.

The mantissa corresponding to 572 is .7574. The difference to be added on account of the final digit 4 in the number is given in the table of differences on the right. In the column under 4, we find that the corresponding difference is 3. This 3 is added to the final digit of the mantissa, giving .7577. It follows that

$$\log 57.24 = 1.7577.$$

Example 3. Find log 824.7.

This number is closer to 825 than to 824. Hence we shall find its mantissa by subtracting from that of 825. Notice that the mantissa of 825 is the same as that of 8250.

For 8250 the mantissa is .9165.

Since 8247 is 3 less than 8250, we subtract from 9165 the difference corresponding to 3. This difference, found from the table, is 2.

Hence $\log 824.7 = 2.9163.$

Writing a negative characteristic. For log 0.4623 the characteristic is seen, by inspection, to be -1 and the mantissa is found, from the tables, to be .6649. Hence

$$\log 0.4623 = -1+.6649.$$

his can be shortened by omitting the + sign and writing the ***over*** the characteristic.

hen

$$\log 0.4623 = \bar{1}.6649.$$

his notation reminds us that the mantissa is always positive id that only the characteristic can be negative.

xample 4. Find (*a*) log 0.7642, (*b*) log 0.005489.

(*a*) Mantissa for 7640 = .8831.
Difference for 2 = 1.
Mantissa for 7642 = .8832.
Characteristic of log 0.7642 = −1.
Hence $\log 0.7642 = \bar{1}.8832.$

(*b*) Mantissa for 5490 = .7396.
Difference for 1 = 1.
Mantissa for 5489 = .7395.
Characteristic of log 0.005489 = −3.
Hence $\log 0.005489 = \bar{3}.7395.$

EXERCISES

id, from the tables, the logarithms of the following numbers:

1. 4.	**2.** 9.	**3.** 80.	**4.** 36.
5. 55.	**6.** 340.	**7.** 790.	**8.** 2600.
9. 6000.	**10.** 235.	**11.** 658.	**12.** 539.
13. 8640.	**14.** 1710.	**15.** 68800.	**16.** 350000.
17. 2431.	**18.** 5283.	**19.** 7644.	**20.** 35.62.
21. 76.29.	**22.** 68.38.	**23.** 574.5.	**24.** 482.6.
25. 8.544.	**26.** 4.376.	**27.** 0.6157.	**28.** 0.5391.
29. 2.009.	**30.** 0.00689.	**31.** 0.00005.	**32.** 0.7007.
33. 0.000277.	**34.** 0.005125.	**35.** 0.09138.	**36.** 0.06666.

How to combine logarithms. In carrying out a computatio it is necessary to perform one or more of the following operation

(1) *adding logarithms;*
(2) *subtracting logarithms;*
(3) *multiplying a logarithm by an integer;*
(4) *dividing a logarithm by an integer.*

Since logarithms are themselves numbers, these operations a carried out by the ordinary rules of arithmetic. When negati characteristics are involved, the operations require some ca as the following examples show.

Example 1. Add the logarithms: $\bar{1}.7462 + 1.6154 + \bar{2}.4286$.

$\bar{1}.7462$
1.6154
$\bar{2}.4286$
$\overline{\bar{1}.7902}$

We leave to you the explanation of t addition.

Example 2. Subtract: $\bar{2}.5283 - \bar{1}.6357$.

With a little practice, you will be able to carry out t subtraction as it is written on the left. If this is clear you, you need not read the explanation given on the rig

$\bar{2}.5283$	$-3 + 1.5283$
$\bar{1}.6357$	$-1 + 0.6357$
$\bar{2}.8926$	$-2 + 0.8926 = \bar{2}.8926$

Example 3. Multiply $\bar{2}.4681$ by 3.

Learn to do such multiplications as in the left-hand colun below. Read the right-hand column only if you need hel

$\bar{2}.4681$	$-2 + .4681$
$\quad 3$	$\quad 3$
$\bar{5}.4043$	$-6 + 1.4043 = \bar{5}.4043$

Example 4. Divide $\bar{1}.3684$ by 4.

You may need to refer to the solution on the right for the explanation of the method. When it is clear to you, you can carry out the division as in the left-hand column.

$4\overline{)\bar{1}.3684}$	$4\overline{)-4 + 3.3684}$
$\bar{1}.8421$	$-1 + 0.8421 = \bar{1}.8421$

EXERCISES

dd:

1. 2.5462	**2.** $\bar{1}.7251$	**3.** $\bar{2}.4685$	**4.** $\bar{3}.6527$
$\bar{1}.4723$	$\bar{1}.5146$	1.7321	$\bar{2}.5146$
	1.2373	$\bar{1}.8139$	2.3752

ubtract:

5. 1.5182	**6.** 0.5728	**7.** 2.1546	**8.** $\bar{3}.4695$
1.9361	$\bar{1}.7654$	$\bar{1}.9251$	$\bar{1}.7521$

Multiply:

9. $\bar{1}.5684$	**10.** $\bar{3}.6135$	**11.** $\bar{2}.5142$	**12.** $\bar{1}.7317$
$\quad 4$	$\quad 2$	$\quad 3$	$\quad 6$

Divide:

13. $2\overline{)\bar{2}.5862}$	**14.** $3\overline{)\bar{1}.4962}$	**15.** $4\overline{)\bar{3}.6152}$	**16.** $5\overline{)\bar{2}.9347}$
17. $2\overline{)\bar{1}.7346}$	**18.** $3\overline{)\bar{2}.5126}$	**19.** $2\overline{)\bar{3}.3965}$	**20.** $4\overline{)\bar{5}.0218}$

Antilogarithms. Consider again the example: $946.2 \div 385.$ (page 293). Using the notation of logarithms, we have

$$\begin{array}{rl} \log\ 946.2 &= 2.9760 \\ \log\ 385.9 &= 2.5865 \\ \hline \text{(subtract)} & 0.3895 \end{array}$$

The final step consists of finding the number whose logarithm i 0.3895. In the foregoing example we had

$$\log\ 2.452 = 0.3895,$$

from which the required number is 2.452.

The number whose logarithm is 0.3895 is called th ***antilogarithm*** of 0.3895. The last equation written abov can also be written antilog $0.3895 = 2.452$.

Similarly, since	$\log\ 1000 = 3,$
therefore	$\text{antilog}\ 3 = 1000,$
and, in general,	$\text{antilog}\ x = 10^x.$

It is possible to find an antilogarithm from a table of logarithm by finding the logarithm in the body of the table and the locating the corresponding number. For greater convenience, ***table of antilogarithms*** is provided (pp. 322 and 323). The us of a table of antilogarithms is illustrated by the followin examples.

Example 1. Find antilog 1.5274.

Consider first only the mantissa .5274.

Referring to the table on page 323, locate .52 in the left-han column and 7 in the top horizontal column. The correspondin entry in the table is 3365. To this we add the difference corre ponding to the final digit 4 of the mantissa. This differenc found in the table at the right of the page, is 3. We now g 3368 as the digits of the number whose logarithm has the ma tissa .5274. It remains to determine the position of the decim point. Since the characteristic of the logarithm is 1, the decim

oint in the number must come after the second significant git. Hence

$$\text{antilog } 1.5274 = 33.68.$$

Notice that, in finding an antilogarithm (that is, in finding number from its logarithm), we *first use the mantissa to determine e significant digits of the number and then use the characteristic fix the position of the decimal point.*

mantissa	*determines*	***significant digits***
characteristic	*determines*	***decimal point***

ample 2. Find antilog $\bar{2}.3426$.

In using the table of antilogarithms, we may *either* find the mber corresponding to the mantissa .342 and then add the ference for 6 *or* find the number corresponding to .343 and otract the difference for 4. These plans are both shown below, ough of course only one need be used.

ntissa .342 → number 2198	mantissa .343 → number 2203
6 → difference 3	4 → difference 2
ntissa .3426 → number 2201	mantissa .3426 → number 2201

The significant digits of the required number are 2201. It nains to fix the position of the decimal point. Since the char- eristic of the logarithm is -2, the decimal point in the number st be in such a position that moving it two places to the ht would bring it after the first significant digit. Hence

$$\text{antilog } \bar{2}.3426 = 0.02201.$$

EXERCISES

1. Given that antilog 0.2846 = 1.926, write the antilog- hms of

1.2846, (*b*) 2.2846, (*c*) $\bar{1}.2846$, (*d*) $\bar{3}.2846$.

2. Find antilog 0.4630.

3. From the result of Ex. 2, write the antilogarithms of (*a*) 1.4630, (*b*) 3.4630, (*c*) $\bar{1}$.4630, (*d*) $\bar{3}$.4630.

4. Find antilog 0.6375.

5. From the result of Ex. 4, write the antilogarithms of (*a*) 1.6375, (*b*) 3.6375, (*c*) 2.6375, (*d*) 4.6375.

Find the numbers which have the following logarithms:

6. 0.7250.	7. 0.6370.	8. 1.4820.	9. 1.5623.
10. 2.8264.	11. 2.5726.	12. $\bar{1}$.3748.	13. $\bar{1}$.1827.
14. $\bar{2}$.4953.	15. $\bar{3}$.6475.	16. $\bar{2}$.9182.	17. $\bar{3}$.3688.
18. $\bar{4}$.4996.	19. $\bar{5}$.2549.	20. $\bar{4}$.7777.	21. 3.8008.
22. 2.0909.	23. $\bar{1}$.7102.	24. 1.0656.	25. $\bar{2}$.1007.

Computation with Logarithms

Multiplication. The law of indices for finding products is

$$10^p \times 10^q = 10^{p+q}.$$

In words this says that the index of the product of powers of 1 is the sum of the indices of these powers. If

$$10^p = M \text{ and } 10^q = N,$$

then $p = \log M$, $q = \log N$, and $p + q = \log MN$.

The foregoing law of indices can now be expressed in symbo and in words as a law of logarithms.

> ***log MN = log M + log N***
> ***The logarithm of a product is the sum of the logarithms of its factors.***

Example 1. Find the product of $32.86 \times 5.461 \times 2.753$.

The computation is written below in two ways. The column n the right is inserted only to remind you of the meaning of the perations. Once you are sure of the use of logarithms, the ıdex forms can be dispensed with.

We denote the product by P.

$$\begin{aligned} \log 32.86 &= 1.5167 \\ \log 5.461 &= 0.7373 \\ \log 2.753 &= 0.4398 \\ \hline \log P &= 2.6938 \\ P &= \text{antilog } 2.6938 \\ &= 494.1 \end{aligned}$$

$$\begin{aligned} 32.86 &= 10^{1.5167} \\ 5.461 &= 10^{0.7373} \\ 2.753 &= 10^{0.4398} \\ \hline P &= 10^{2.6938} \\ &= 494.1 \end{aligned}$$

It should be remembered that the answer is correct only to ur figures.

EXERCISES

ɔmpute the following products:

1. 75.24×1.832.
2. 6.786×5.923.
3. 8462×0.5467.
4. 0.6483×0.4526.
5. 0.7527×0.03821.
6. $8.357 \times 9.625 \times 2.861$.
7. $2.463 \times 5.386 \times 28.56$.
8. $17.24 \times 3.759 \times 8.642$.
9. $0.4531 \times 2.783 \times 0.6945$.
10. $6.230 \times 7.542 \times 8.764 \times 3.175$.
11. $73.26 \times 0.1723 \times 5.835 \times 0.04920$.
12. $0.4571 \times 0.6482 \times 0.009264 \times 0.0003563$.
13. Find the area of a rectangle whose sides are 31.56 cm. d 54.24 cm. Is your answer correct to the nearest square cm. to the nearest tenth of a square cm.?
14. Find the volume of a rectangular solid whose edges are .45 cm., 11.68 cm., and 15.23 cm. Indicate the degree of curacy of your result.

15. Compute the weight of a rectangular plate 28.5 inch long, 16.9 inches wide, and 3.57 inches thick, if the metal weigh 0.254 pounds per cubic inch.

16. The area of a cylinder is given by the formula $2\pi rl$. Fin the area of a cylinder whose radius is 16.25 inches and leng 10.80 inches. (Take $\pi = 3.142$.)

Division. The index law for division states that

$$10^p \div 10^q = 10^{p-q}.$$

The index of the quotient of two powers of 10 is equal to t index of the dividend minus the index of the divisor. If

$$10^p = M \text{ and } 10^q = N,$$

then $p = \log M$, $q = \log N$, and $p - q = \log \frac{M}{N}$.

The foregoing index law becomes the following rule of logarithm

$$\log \frac{M}{N} = \log M - \log N$$

The logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.

Example 2. Find the value of $34.77 \div 872.5$.

If the solution on the left is clear to you, you need not re the explanation of it on the right.

Denote the quotient by Q.

$$\begin{array}{l} \log 34.77 = 1.5412 \\ \log 872.5 = 2.9408 \\ \hline \log Q = \bar{2}.6004 \\ Q = \text{antilog } \bar{2}.6004 \\ \quad = 0.03985 \end{array}$$

$$\begin{array}{l} 34.77 = 10^{1.5412} \\ 872.5 = 10^{2.9408} \\ Q = \dfrac{34.77}{872.5} = \dfrac{10^{1.5412}}{10^{2.9408}} = 10^{1.5412-2.940} \\ \qquad = 10^{\bar{2}.6004} \\ \log Q = \bar{2}.6004 \\ Q = 0.03985 \end{array}$$

EXERCISES

Compute the values of the following:

1. $65.43 \div 21.46$.
2. $721.6 \div 13.24$.
3. $8.652 \div 23.65$.
4. $8523 \div 6194$.
5. $0.4768 \div 7.233$.
6. $0.06542 \div 0.4721$.
7. $5.483 \div 78.16$.
8. $9.651 \div 0.06437$.
9. $0.05870 \div 0.8265$.
10. $2.944 \div 300.7$.
11. $80.09 \div 0.6073$.
12. $0.006523 \div 4.8257$.
13. $0.0008264 \div 0.02368$.
14. $0.00003333 \div 0.007777$.

15. The area of a rectangle is 48.26 sq. in. and one side is 485 in. Find the other side.

16. The volume of a marble column is 347.5 cu. ft. and the base contains 4.270 sq. ft. Find the height of the column.

17. Find the height of a triangle, given that its area is 348.2 sq. in. and its base is 25.36 in.

18. The density of gold is 19.34 gm. per cu. cm. Find the volume of a gold ingot which weighs 254.6 gm.

19. Find the average speed of a point which travels 682.5 ft. in 38.90 minutes.

Powers. We take now the index law

$$(10^p)^r = 10^{rp}.$$

$$10^p = M \text{ so that } M^r = 10^{rp},$$

then $\log M = p$, and $\log M^r = rp = r \log M$.

This gives the following important rule of logarithms.

> ***log $M^r = r$ log M***
> ***The logarithm of a power of M is equal to the logarithm of M multiplied by the index of the power.***

Hence, by the use of logarithms, the finding of a power is r
placed by a simple multiplication.

Example 3. Find the value of $(3.265)^5$.

The solution is given below in two forms. Since the purpose
logarithms is to save time and labour, you should use the for
on the left as soon as its meaning is clear to you.

Set $N = (3.265)^5$	$\log 3.265 = 0.5139$
$\log 3.265 = 0.5139$	$N = (3.265)^5 = (10^{0.5139})^5$
$\underline{\qquad\qquad 5}$	$= 10^{2.5695}$
$\log N = 2.5695$	$\log N = 2.5695$
$N = 371.1$	$N = 371.1$

EXERCISES

Find the values of the following powers:

1. $(7.547)^2$.
2. $(4.621)^3$.
3. $(2.564)^4$.
4. $(58.81)^2$.
5. $(0.4253)^5$.
6. $(0.08234)^6$.
7. $(0.004646)^3$.
8. $(0.008968)^4$.
9. $(0.7327)^8$.
10. Given that $\log 2 = 0.30103$, calculate 2^{10}.
11. Given that $\log 3 = 0.47712$, calculate 3^8.
12. Find the area of a square whose side is 58.66 in.
13. Find the volume of a cube whose edge is 9.267 cm.

Roots. The index law which applies to roots is

$$\sqrt[r]{10^p} = 10^{p/r}.$$

If $10^p = M$, from which $\sqrt[r]{M} = 10^{p/r}$,

then $\log M = p$ and $\log \sqrt[r]{M} = p/r = \frac{1}{r} \log M$.

This gives another important rule of logarithms.

$$\log \sqrt[r]{M} = \frac{1}{r} \log M$$

The logarithm of the rth root of a number is equal to $\frac{1}{r}$ times the logarithm of the number.

The importance of logarithms for finding roots is due not erely to the fact that the time and labour involved is small, ut also to the fact that, for most roots, there is no other simple ethod of finding them.

xample 4. Find the value of $\sqrt[3]{0.4853}$.

s before, two solutions are given. The form on the left is the ıe in which the logarithms fulfil their purpose of saving time ıd labour.

Set $N = \sqrt[3]{0.4853}$.

$\log 0.4853 = \bar{1}.6860$	$\log 0.4853 = \bar{1}.6860$
$3 \overline{)\bar{1}.6860}$	$N = \sqrt[3]{10^{\bar{1}.6860}} = 10^{\bar{1}.6860/3}$
$\log N = \bar{1}.8953$	$= 10^{\bar{1}.8953}$
$N = 0.7857$	$N = 0.7857$

EXERCISES

ɔmpute the following roots:

1. $\sqrt{8645}$.
2. $\sqrt{37.26}$.
3. $\sqrt[3]{56.58}$.
4. $\sqrt[5]{283.7}$.
5. $\sqrt{0.4623}$.
6. $\sqrt[3]{0.05821}$.
7. $\sqrt[4]{0.08462}$.
8. $\sqrt[6]{7.564}$.
9. $\sqrt[5]{0.006254}$.
10. Find the side of a square whose area is 86.53 sq. in.
11. Find the edge of a cube whose volume is 647.8 cu. in.

The further use of logarithms. It is now clear that, whenever a computation involves only products, quotients, powers and roots of numbers with several digits, labour can be saved by using logarithms. *It should be noted that logarithms are not adapted to computing sums or differences.*

In computing a complicated expression, it is necessary to be able to express the logarithm of the expression in simple form before looking up the tables.

The following examples give practice in doing this.

Example 5. Express in simple form:

$$(a)\ \log \frac{MN}{P};\quad (b)\ \log ab^2c^3;\quad (c)\ \log \frac{r^2}{\sqrt{s}}.$$

In the following solutions the parts in brackets may be omitted if the results are clear without them.

$$(a)\ \log \frac{MN}{P} = (\log MN - \log P) = \log M + \log N - \log P$$

$$(b)\ \log ab^2c^3 = (\log a + \log b^2 + \log c^3)$$
$$= \log a + 2 \log b + 3 \log c.$$

$$(c)\ \log \frac{r^2}{\sqrt{s}} = (\log r^2 - \log \sqrt{s}) = 2 \log r - \tfrac{1}{2} \log s.$$

EXERCISES

Express the following in simple form:

1. $\log (ab)^3$.
2. $\log \left(\frac{p}{q}\right)^4$
3. $\log 2\pi rh$.
4. $\log \frac{1}{3}\pi r^2 h$.
5. $\log \pi\sqrt{l/g}$.
6. $\log \frac{M^2}{NR}$.
7. $\log \sqrt{MN^3}$.
8. $\log \frac{M^2\sqrt{N}}{R}$

Express the following in terms of log 2 and log 3:

9. log 8.
10. log 81.
11. log 36.
12. $\log \frac{16}{27}$.

Further computation—and some advice. In computing with ogarithms, it is not sufficient to understand the principles in- olved. It is necessary also to acquire skill sufficient to avoid rrors and to save time. Of the two objectives (1) avoiding rrors and (2) saving time, the former is much the more im- ortant. You should aim to achieve it first; the saving of time ill then come with practice. Two bits of advice may be elpful:

(1) *Do* ***all*** *your computation on the page on which you write our solution, in order that every step can be checked;*

(2) *When the tables have to be used several times, make a skeleton f your solution before looking up logarithms. Then look up all the garithms successively before proceeding further.* In this way you re less likely to make errors in the use of tables.

xample 6. Find the area of the circle with radius 23.48 cm.

We use the formula $A = \pi r^2$ and take $\pi = 3.142$.

We have

$$\log A = \log \pi + 2 \log r.$$

$$\begin{aligned} \log 23.48 &= 1.3707 \\ &\quad\ \ 2 \\ &\overline{\quad 2.7414} \\ \log 3.142 &= 0.4972 \\ \log A &= 3.2386 \\ A &= 1732. \end{aligned}$$

Hence the area is 1732 sq. cm.

xample 7. Compute the value of $\dfrac{(2.465)^2 \times 64.23}{\sqrt{523.4} \times 8.362}$.

Denote this number by N. We have

$$\log N = 2 \log 2.465 + \log 64.23 - \tfrac{1}{2} (\log 523.4 + \log 8.362).$$

)ne way of saving time, when you are proficient in the use of garithms, is by omitting this line.)

In the following solution, the column on the left is inserted only to show the skeleton which is constructed before the logarithms are looked up. The completed solution is given on the right.

$\log 2.465 = 0.$	$\log 2.465 = 0.3918$
$\underline{\qquad\qquad 2}$	$\underline{\qquad\qquad 2}$
	0.7836
$\log 64.23 = 1.$	$\log 64.23 = 1.8077$
	$2.5913 = 2.5913$
$\log 523.4 = 2.$	$\log 523.4 = 2.7188$
$\log 8.362 = 0.$	$\log 8.362 = 0.9223$
$2\boxed{\qquad}$	$2\boxed{3.6411}$
	$= 1.8205 = 1.8205$
$\log N =$	$\log N =$ $\qquad 0.7708$
$N =$	$N = 5.899.$

EXERCISES

Carry out the following computations with logarithms:

1. $6.754 \times 8.736 \times 5.270.$

2. $\dfrac{78.46 \times 3.752}{25.94}.$

3. $\dfrac{31.57 \times 5.826}{7.459 \times 46.72}.$

4. $(5.263)^2 \times 17.66.$

5. $72.35 \times \sqrt{9680}.$

6. $\sqrt{\dfrac{82.56}{3.712}}.$

7. $6875 \times (1.055)^3.$

8. $36.45 \times \sqrt[3]{236.8}.$

9. $\sqrt{23.44 \times (5.260)^3}.$

10. $\dfrac{65.37 \times (4.783)^3}{48.65 \times (7.814)^2}.$

11. $\dfrac{(16.52)^2 \times 28.32}{\sqrt{254.8}}.$

12. $\dfrac{\sqrt[3]{0.06378}}{\sqrt[5]{0.004516}}.$

13. $\dfrac{(3.715)^2 \times \sqrt{0.8236}}{\sqrt[3]{0.04854}}$. 14. $\sqrt{\dfrac{0.4861 \times (0.8234)^3}{0.09263 \times 0.7257}}$.

aking $\pi = 3.142$, compute the following:

15. The area of a circle with radius 23.45 cm.

16. The volume of a cylinder ($\pi r^2 l$) with radius 4.825 cm. nd length 75.23 cm.

17. The volume of a sphere ($\frac{4}{3}\pi r^3$) with radius 5.720 cm.

18 The volume of a spheroid ($\frac{4}{3}\pi a^2 b$) in which $a = 12.35$ nd $b = 5.420$.

EXERCISES ON CHAPTER XIII

A

1. Give (*a*) the definition of the logarithm of a number, (*b*) ie name of the integral part of a logarithm, (*c*) the name of the actional part.

2. Change the following equations into logarithmic notation:

(*a*) $10^2 = 100$, (*b*) $10^{-2} = 0.01$, (*c*) $10^0 = 1$,
(*d*) $10^1 = 10$, (*e*) $10^{0.30103} = 2$, (*f*) $10^l = n$.

3. Change the following equations into index notation:

(*a*) log 1000 = 3, (*b*) log 5 = 0.69897,
(*c*) log 30 = 1.47712.

4. Give the values of:

(*a*) log 10000, (*b*) log 1 000 000, (*c*) log 1, (*d*) log 0.1, (*e*) log 0.001, (*f*) log 0.00001.

5. Give the characteristics of the logarithms of:

(*a*) 37468, (*b*) 3.1416, (*c*) 0.052, (*d*) 10.02, (*e*) 1728.5, (*f*) 0.0008.

6. Find, from the tables, the logarithms of:

(*a*) 7.49, (*b*) 643, (*c*) 8, (*d*) 34.62, (*e*) 746.8, (*f*) 0.5555.

7. Find from the tables the numbers whose logarithms are

(*a*) 1.5466, (*b*) 0.3275, (*c*) 2.7158, (*d*) $\bar{1}.5264$, (*e*) $\bar{2}.239$
(*f*) $\bar{3}.6413$.

Use logarithms to compute the following:

8. (*a*) 76.43×3.287, (*b*) 183.4×4.256.

9. (*a*) $382.6 \div 54.37$, (*b*) $7925 \div 526.9$.

10. (*a*) $(18.25)^2$, (*b*) $(4.213)^3$.

11. (*a*) $\sqrt{28.54}$, (*b*) $\sqrt{0.7234}$.

12. (*a*) $\sqrt[3]{0.006743}$, (*b*) $\sqrt[3]{0.07582}$.

B

13. Express in terms of logarithms of simpler numbers:

(*a*) log 25, (*b*) log 81, (*c*) log 32,
(*d*) log $\frac{5}{3}$, (*e*) log $\frac{8}{9}$, (*f*) log $\frac{121}{128}$,
(*g*) log $\sqrt{5}$, (*h*) log $\sqrt[3]{16}$, (*i*) log $8\sqrt{3}$.

14. Combine each of the following into a single logarithm term:

(*a*) log 2 + log 3 + log 4, (*b*) log 9 + log 4 − log 6.
(*c*) 2 log 3 + $\frac{1}{2}$ log 25, (*d*) 2 log 7 + 3 log 2 − log 28,
(*e*) $\frac{1}{4}$ log 16 + $\frac{1}{3}$ log 125, (*f*) 2 log 5 − log 15 + 3 log 3.

15. Express as sums and differences of simpler terms:

(*a*) log $\frac{1}{2}bh$, (*b*) log $4\pi r^2$, (*c*) log $2\pi^2 ab^2$,
(*d*) log $\frac{4}{3}\pi r^3$, (*e*) log $\frac{1}{2}mv^2$, (*f*) log $\sqrt{s(s-a)(s-b)(s-c)}$.

Use logarithms to compute the following:

16. $9.348 \times 17.52 \times 6.893$.

17. $\dfrac{45.26 \times 231.5}{322.8}$.

18. $\dfrac{1.175 \times 2.175}{3.175 \times 4.175}$.

19. $\frac{1}{2}(56.47) \times (2.450)^2$.

20. $\dfrac{2.576\sqrt{92.58}}{0.8264}$.

21. $\dfrac{1.726^2 \times 2.543^3}{5.281}$.

22. $\dfrac{82.55 \times 1.264}{\sqrt{7.160 \times 11.05}}$.

23. $\dfrac{3.562 \times (7.481)^2}{8.754 \times (2.568)^3}$.

24. $\sqrt{\dfrac{89.52 \times 6.437}{5.296 \times 48.53}}$.

25. $3.216\sqrt[4]{(4.735)^3}$.

26. $\dfrac{\sqrt[3]{(8.422)^2}}{5.478}$.

27. $\dfrac{(0.7465)^2\sqrt{0.5147}}{(0.3274)^3\sqrt[3]{0.8942}}$.

28. Find the volume of a rectangular solid whose dimensions e 4.268 in., 9.432 in., and 5.876 in.

29. Find the area of a circle with radius 23.42 ft.

30. Find the volume of a sphere with radius 8.260 in.

31. Find the curved surface of a cylinder with radius 3.54 . and height 25.5 in.

32. Find the radius of a sphere whose volume is 678.4 cu. in.

33. Find the radius of a right circular cylinder whose volume 365.2 cu. in. and length 27.24 in.

34. The number of seconds for a complete oscillation of a nple pendulum L cm. long is given by $T = 2\pi\sqrt{L/g}$, where = 980. Find T for a pendulum 82.4 cm. long.

35. The area of a triangle is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

here a, b, and c are the measures of the sides of the triangle and of half the perimeter. Find the area of a triangle whose sides e 32.57 yards, 58.24 yards, and 67.94 yards, respectively.

No.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6

0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Log.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6

0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

degrees	sine	cosine	tangent	cotangent	degrees
0	.000	1.000	.000	.....	90
1	.017	1.000	.017	57.29	89
2	.035	.999	.035	28.64	88
3	.052	.999	.052	19.08	87
4	.070	.998	.070	14.30	86
5	.087	.996	.087	11.43	85
6	.105	.995	.105	9.51	84
7	.122	.993	.123	8.14	83
8	.139	.990	.141	7.12	82
9	.156	.988	.158	6.31	81
10	.174	.985	.176	5.67	80
11	.191	.982	.194	5.14	79
12	.208	.978	.213	4.70	78
13	.225	.974	.231	4.33	77
14	.242	.970	.249	4.01	76
15	.259	.966	.268	3.73	75
16	.276	.961	.287	3.49	74
17	.292	.956	.306	3.27	73
18	.309	.951	.325	3.08	72
19	.326	.946	.344	2.90	71
20	.342	.940	.364	2.75	70
21	.358	.934	.384	2.61	69
22	.375	.927	.404	2.48	68
23	.391	.921	.424	2.36	67
24	.407	.914	.445	2.25	66
25	.423	.906	.466	2.14	65
26	.438	.899	.488	2.05	64
27	.454	.891	.510	1.96	63
28	.469	.883	.532	1.88	62
29	.485	.875	.554	1.80	61
30	.500	.866	.577	1.73	60
31	.515	.857	.601	1.66	59
32	.530	.848	.625	1.60	58
33	.545	.839	.649	1.54	57
34	.559	.829	.675	1.48	56
35	.574	.819	.700	1.43	55
36	.588	.809	.727	1.38	54
37	.602	.799	.754	1.33	53
38	.616	.788	.781	1.28	52
39	.629	.777	.810	1.23	51
40	.643	.766	.839	1.19	50
41	.656	.755	.869	1.15	49
42	.669	.743	.900	1.11	48
43	.682	.731	.933	1.07	47
44	.695	.719	.966	1.04	46
45	.707	.707	1.000	1.00	45
degrees	cosine	sine	cotangent	tangent	degrees

TABLE OF SQUARES AND SQUARE ROOTS

n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$
1	1	1.000	16	256	4.000	31	961	5.568	46	2116	6.782
2	4	1.414	17	289	4.123	32	1024	5.657	47	2209	6.856
3	9	1.732	18	324	4.243	33	1089	5.745	48	2304	6.928
4	16	2.000	19	361	4.359	34	1156	5.831	49	2401	7.000
5	25	2.236	20	400	4.472	35	1225	5.916	50	2500	7.071
6	36	2.449	21	441	4.583	36	1296	6.000	51	2601	7.141
7	49	2.646	22	484	4.690	37	1369	6.083	52	2704	7.211
8	64	2.828	23	529	4.796	38	1444	6.164	53	2809	7.280
9	81	3.000	24	576	4.899	39	1521	6.245	54	2916	7.348
10	100	3.162	25	625	5.000	40	1600	6.325	55	3025	7.416
11	121	3.317	26	676	5.099	41	1681	6.403	56	3136	7.483
12	144	3.464	27	729	5.196	42	1764	6.481	57	3249	7.550
13	169	3.606	28	784	5.292	43	1849	6.557	58	3364	7.616
14	196	3.742	29	841	5.385	44	1936	6.633	59	3481	7.681
15	225	3.873	30	900	5.477	45	2025	6.708	60	3600	7.746

Set. No. 1. Pages 2-4

1. -1. 2. 3. 3. 1. 4. 5.
5. -3. 6. 0.16. 7. 9. 8. -5.256.
9. 0.36. 10. 0. 11. 21. 12. -0.08.
3. 0. 14. -13. 15. 5.8. 16. $5\frac{2}{3}$.
7. 17.4. 18. 3.4. 19. 7.84. 20. 4.144.
1. -15.5. 22. 1. 23. $-5x$. 24. $-9.4ab$.
5. $-\frac{1}{4}m$. 26. $5m^2$. 27. $-13xy$. 28. $3.6a^2b^2$.
9. $-2x$. 30. $-0.7pq$. 31. $0.45ab$. 32. $-2x+1.2$.
3. $-8x^2-4y^2z$. 34. $-a^2-8b^2+10$. 35. $x^2-10x+9$.
6. $17a^2-9a-1$. 37. $-5.5z^2+14.7z-9$.
8. $2m^2-3m-3$; $4m^2-13m+21$.
9. $13a-2b-3$; $a+8b-7$.
0. $21c+d-2e$; $3c-17d+8e$.
1. $-3x^2+11x+6$.
2. $-3a^2-11b^2+8ab$. 43. $3y$. 44. $-7mn+kn-5km$.
5. x^2-3. 46. $-3y$. 47. $9r^3-8r^2-8s^3$.

Set No. 2. Pages 6-7

1. $m^2+2mn+n^2$. 2. $a^2-2ax+x^2$. 3. c^2-d^2.
4. $9m^2-12mn+4n^2$. 5. $x^3+3x^2y+3xy^2+y^3$.
6. $x^3-6x^2y+12xy^2-8y^3$. 7. $4k^2+20km+25m^2$.
8. $m^2+n^2+r^2+2mn+2nr+2mr$.
9. $49x^2-42xy+9y^2$. 10. $49x^2-9y^2$.
1. $x^2+4y^2+z^2-4xy+4yz-2xz$. 12. $9x^4-81y^4$.
3. $4a^2+b^2+9c^2-4ab-12ac+6bc$.
4. $27c^3+54c^2d+36cd^2+8d^3$.
5. $64m^3-144m^2n+108mn^2-27n^3$.
6. $6x^3-29x^2+43x-20$.
7. $15a^4-a^3b-19a^2b^2+9ab^3-4b^4$.
8. $6x^4+11x^3y-12x^2y^2+33xy^3-20y^4$.
9. $x^3+y^3+z^3-3xyz$. 20. $27x^3-64y^3$.

21. $-2x^2 + xy - 3y^2$. **22.** $2a + 3b - a^2$.
23. $9a^2 + 9a + 5$. **24.** $x^2 + 1$.
25. $3x^2 - 2x + 1$. **26.** $a^2 + a + 1$.
27. $x^2 - 2xy - 3y^2$. **28.** $-m^2 - 3m - 1$.
29. $z^2 + 3z + 2$. **30.** $x^2 + xy + y^2$.
31. $a^2 - ab + b^2$. **32.** $2x^2 - 2xy + 4y^2$.
33. $2ac + 4bc - b^2$. **34.** $6a^2b + 2b^3$.
35. $-7x^2 - 36xy + 10y^2$. **36.** $4m^2 + 48mn + 11n^2$.
37. $-22x^2 + 37x + 1$. **38.** $2x^3 - 32x^2 + 46x + 4$.
39. $16a^2 + 20a - 60$. **40.** $6a^2 + 4b^2 - 2c^2$.
41. $-41x^3 + 30x^2 - 51x + 74$. **42.** $-38xy + 72y^2$.
43. $330x - 3067$. **44.** $-87x^3 - 148x^2y - 202xy^2 - 191y$

Set No. 3. Pages 9-11

1. 2. **2.** 3. **3.** -2. **4.** 15.
5. 3. **6.** -5. **7.** -8. **8.** 0.3.
9. b/a. **10.** $n - m$. **11.** 5. **12.** 1.
13. 3.1. **14.** -4. **15.** 1. **16.** -3.
17. 10. **18.** 14. **19.** 1. **20.** 24.
21. -3. **22.** $-\frac{1}{2}$. **23.** 4. **24.** $-\frac{1}{2}$.
25. 7. **26.** -2. **27.** 24. **28.** 5.
29. -6. **30.** 55. **31.** -2. **32.** 0.
33. -1. **34.** 4. **35.** $x = 10, y = 3$.
36. $x = 1, y = 4$. **37.** $m = 1, n = \frac{1}{2}$. **38.** $a = 3, b = -2$
39. $x = 2, y = 3$. **40.** $x = 1, y = 0.5$. **41.** $x = 3, y = 2$.
42. $x = 1, y = -1$. **43.** $x = 0.3, y = -0.2$.
44. $x = \frac{1}{2}, y = -\frac{1}{2}$. **45.** $x = 1, y = 2$. **46.** $x = \frac{1}{2}, y = \frac{1}{5}$.

Set No. 4. Pages 13-16

1. 22. **2.** 212. **3.** 2870. **4.** 160.
5. 600. **6.** 630. **7.** About 115. **9.** $V = lwh$.
10. $V = \frac{1}{3}\pi r^2 h$. **12.** $d = 16t^2$. **14.** $d = rt$. **15.** $h = V/lw$.
16. $x = \pm\sqrt{(K - a)/b}$. **17.** 5. **18.** 50.
19. 3.5 in. **20.** 7 in. **21.** 6.

2. p is (a) increased 4-fold; (b) reduced to 1/9 of its original value; (c) increased to 9/4 times its original value.
3. Cost is (a) increased 8-fold; (b) increased 27-fold; (c) reduced to 1/8 of its original value.
4. W is (a) halved; (b) reduced to 1/3 of its original value; (c) increased 4-fold.
5. $2a(2 + \pi)$; $a^2(4 - \pi)$.
7. $4\pi m$; $m^2(4 + 2\pi)$.
9. $y = 2.5x$.
1. (a) 73.5 ft. (b) 228 ft.
3. (a) 132 lb.; (b) 176 lb.
26. $2\pi r$; $\frac{1}{2}\pi r^2$.
28. $3\pi x$; $x^2(6 - \frac{5}{4}\pi)$.
30. $c = 2w + 2$.
32. (a) $A = 6x^2$; (b) $V = x^3$; (c) a 4-fold increase; (d) an 8-fold increase.

Set No. 5. Pages 19-20

6. $x + 12$; $x - 12$; $12 - x$; $x/12$; $5 + 3x$.
7. 17.
18. 25.
19. 18, 14.
0. 10 in.
21. 38°, 76°, 66°.
22. 35 rods by 25 rods.
3. 17.
24. 48, 49.
25. $1020.00.
6. 40°, 140°, 40°, 140°.
27. 200 threes, 100 fours.
8. 12, 93.
29. 105 nickels, 35 dimes.
0. 60 lb. @ 50c.; 40 lb. @ 60c.
31. $3\frac{1}{2}$ miles.
2. 5.6 ft., 22.4 ft.
33. 40, 41, 42, 43.
4. 80°, 80°, 20°.
35. 30 miles.

Pages 52-53

9. $\frac{3}{4}$.
20. $2/3a$.
21. $x + y$.
2. 7/9.
23. $2n/3m$.
24. $\dfrac{a - 1}{a + 1}$.
5. (a) $\frac{1}{5}$, (b) $\frac{1}{4}$, (c) $\frac{2}{5}$, (d) $\frac{3}{4}$, (e) $\frac{2}{3}$.
6. (a) 5 : 6, (b) 22 : 7, (c) 19 : 20.
9. -8.
31. 100, 260.
29. 10.
32. (a) 300, 240, (b) 243, 297, (c) 160, 380.

Pages 55-56

1. $\frac{5}{7}$.
2. $\frac{5}{2}$.
3. $\frac{1}{4}$.
4. $\frac{2}{3}$.
5. $\frac{4}{1}$.
6. $\frac{1}{8}$.
7. $\frac{3}{4}$.
8. $\frac{22}{15}$.
9. $\frac{3}{10}$.
10. \$40.08, \$50.10, \$60.12.
11. \$181.45, \$254.03.
12. 64 rods by 48 rods.
13. (*a*) 1.219/1, (*b*) 1.609/1.
14. (*a*) $\frac{34}{75}$, (*b*) $\frac{34}{25}$.
15. $166\frac{2}{3}$ ft.
16. 125 : 64, $\frac{64}{125}$, 61:64.
17. 3 : 1.
18. 1 : 1.
19. 1 : 3.

Pages 58-59

1. 2.4.
2. -44.
3. -21.
4. 6.
5. ± 4.
6. -8.
7. 11.5.
8. 2.1.
9. $1\frac{1}{2}$.
10. (*a*) nr/s, (*b*) nr/m.
11. xd/c; 5.
12. (*a*) 4 : 3, (*b*) 3 : (*c*) $-2 : 7$.
14. 3.
15. 4.
16. (*a*) ± 8, (*b*) ± 1.5, (*c*) $\pm a^3b^2$, (*d*) $\pm (x^2 - y^2)$.

Page 61

1. $x = 15$, $y = 21$.
2. $x = 2.8$, $y = 8$.
3. 37 in.
4. 76 ft.

Pages 66-67

22. 150 ft. (2 figs.)
23. 15 ft. (2 figs.)
24. 104 yd. (2 figs.)
25. 490 yd. (2 figs.)
26. 75 ft.
27. 310 yd. (2 figs.
28. 420 ft. (2 figs.)
29. 150 ft. (2 figs.)
30. 500 ft. (2 figs.)

Pages 69-70

6. 2.5.
7. (*a*) 4, (*b*) 24.
8. 2.6 cm. (2 figs.)
9. About 20 gals.
10. \$45.83.
11. (*a*) $\frac{22}{7}$, (*b*) 264 i
12. 124 volts.
13. 37.5 cm.
14. 40.
15. 30 ft.

Pages 73-74

6. 15. 7. (*a*) 7.7; (*b*) 5.5. 8. 2.5.
9. 96 sq. in.; 6 in. 10. (*a*) 50, (*b*) 240, (*c*) 75, (*d*) 50.
1. (*a*) 240 vib./sec.; (*b*) 60 cm.
2. (*a*) About 167 vib./sec.; (*b*) $\frac{1}{8}$ cm.
3. (*a*) 25 amp.; (*b*) $3\frac{1}{8}$ ohms.
4. (*a*) 4000 cu. ft.; (*b*) About 267 cu. ft.

Review Exercises on Chapter III. Pages 74-77

4. 20, 35. 5. 160, 180. 7. \$75, \$105, \$135.
8. (*a*) 4 : 3, (*b*) 5 : 6, (*c*) 2 : 1, (*d*) 2 : 1. 9. 10.
0. 4. 11. 4. 12. $\frac{1}{4}$.
3. 0.57, 0.82, 0.70. 15. 5. 16. 10.
7. 4. 18. 200 ft. 19. (*a*) ± 15, (*b*) $\pm x^3y^2$.
0. 5. 21. 1. 22. $a = 4.5$, (*b*) $= 3\frac{1}{3}$
3. 72 ft. 24. (*a*) 5 : 2, (*b*) 6 in., (*c*) 3.6 in.
5. 260 ft. (2 figs.) 26. 51 ft. (2 figs.)
7. 31 ft. (2 figs.) 28. 16.
9. $12\frac{2}{3}$ m.p.h. 30. 7.6.
1. 232.5; 7.5. 32. About 115 ft.
3. 62°; 28 ft. (2 figs.) 34. 12.7.
6. 1 : 4 : 9. 37. \$2.30, \$3.68.
8. 9 and 25. 39. 3 : 2.
0. $\frac{5}{8}$. 41. 15, 18.
2. \$72.

Pages 84-86

1. $a^2 + 2ab + b^2 - c^2$. 22. 9991.
3. $x^2 - 4xy + 4y^2 - 9z^2$. 24. $m^2 - n^2 + 4kn - 4k^2$.
5. $9x^2 - y^2 - 2yz - z^2$. 26. 3575.
7. $r^2 + 2rs + s^2 - 4t^2$. 28. $25x^2 - y^2 - 2yz - z^2$.
9. $c^2 + 2cd + d^2 - b^2$. 30. $(2x - 1)(x + 1)$.
1. $2(x - 2)(x + 2)$. 32. $z(z - 1)(z + 1)$.
3. $2b(b - 3)(b + 1)$. 34. $a(x + 2)(x + 3)$.

35. $x^2(x-1)(x+1)$.
36. $3y^2(y-2)(y+2)$.
37. $mx^2(x-1)$.
38. $4(x-5)(x-3)$.
39. $(3m+1)(m-1)$.
40. $(x-\frac{1}{2})(x+\frac{1}{2})$.
41. $(n-6)^2$.
42. $(3xy-1)(3xy+1)$.
43. $5(2x-1)(x+2)$.
44. $(3x+2)(x+5)$.
45. $(3a-1)(4a+3)$.
46. $(8x-7)(x+1)$.
47. $(1-2km)(1+2km)$.
48. $12a(3b-a-2c)$.
49. $(x-1)(x+1)(x^2+1)$.
50. $(3-y)(3+y)(9+y^2)$.
51. $xy(x+1)^2$.
52. $2(a-13)(a+2)$.
53. $(3x+4)(2x-1)$.
54. $3(2x-5)(x+2)$.
55. $k(m-5)(m+5)$.
56. $(12x-y)(x+9y)$.
57. $(4x-9y)(3x-y)$.
58. $(6x-y)(2x+9y)$.
59. $(5a-4b)^2$.
60. $(m+n)(x+y)$.
61. $(m+n)(x-y)(x+y)$.
62. $(c+d-n)(c+d+n)$.
63. $(k-m-n)(k+m+n)$.
64. $(p+q)(a+b-1)$.
65. $(2x-3y-5)(2x-3y+5)$.
66. $(x+2)(x+3)(x^2+5x+5)$.
67. $(3x+4)(x-2)(3x+1)(x-1)$.
68. $(2x-5y)(x+3y)$.
69. $(a+b-c+d)(a+b+c-d)$.
70. $(5a+2b)(3a-4b)$.
71. $(x-y)(x-y-5)$.
72. $(m+n)^2(m+n-1)(m+n+1)$.
73. $(3x-8y)(2x-3y)$.
74. $5x^2+13x-6$.
75. $8a^2-34ab+21b^2$.
76. (*a*) $2x^2-14x+3$; (*b*) $20a(3c-2b)$.

Pages 87-88

1. $(x+y)(c+d)$.
2. $(k-m)(4+a)$.
3. $(x+y)(p-q)$
4. $(m-n)(a-b)$.
5. $(x-2)(3x^2-5)$.
6. $(x-1)^2(x+1)$.
7. $(y+m)(k-y)$.
8. $(x-w)(7y-3z)$.
9. $x(x+1)$.
10. $(b-1)(a+1)$.
11. $(x+y-z)(x+y+z)$.
12. $(2m-n-3k)(2m-n+3k)$.
13. $(a-b-c)(a+b+c$
14. $(5p-q+3)(5p+q-3)$.
15. $(1-x+y)(1+x-y)$.
16. $(4x-3y+5z)(4x+3y-5z)$.

7. $(6n - m + 4)(6n + m - 4)$.
8. $(a + b - c - d)(a + b + c + d)$.
9. $(2x - 3 - y + 3a)(2x - 3 + y - 3a)$.
10. $(p + 1 - q + 4)(p + 1 + q - 4)$.
11. $(a + 5b)(a - 5b - 3)$.
12. $(x - 2y)(p + q - 1)$.
13. $(x - y)(x - y + 1)$.
14. $(2k + n)(2k + n - 2)$.
15. $(3a - 2b)(3a + 2b + 15)$.
16. $(x + 3)(x - 1)(x + 1)$.
17. $(5x - y)(5x - y - a - 1)$.
18. $5(m - n - 3k)(m - n + 3k)$.
19. $(2p - q)(2p + q)(x - 3y)(x + 3y)$.
20. $(a - b)(7 + b)$.

Page 90

1. $(m^2 + mn + n^2)(m^2 - mn + n^2)$.
2. $(p - 1)(p + 1)(p - 2)(p + 2)$.
3. $(x^2 + x + 2)(x^2 - x + 2)$.
4. $(y + 1)(y - 1)(y + 3)(y - 3)$.
5. $(2m^2 + 2m + 1)(2m^2 - 2m + 1)$.
6. $(2c^2 + c + 1)(2c^2 - c + 1)$.
7. $(4x^2 + 2xy + 5y^2)(4x^2 - 2xy + 5y^2)$.
8. $(7k^2 + 3km - 3m^2)(7k^2 - 3km - 3m^2)$.
9. $(a^2 + 4ab - b^2)(a^2 - 4ab - b^2)$.
10. $(a^2 + 4ab + b^2)(a^2 - 4ab + b^2)$.
11. $(5x + 8y)(x - y)(5x - 8y)(x + y)$.
12. $(5x^2 + 6xy + 8y^2)(5x^2 - 6xy + 8y^2)$.
13. $25(2p^2 + 2p + 1)(2p^2 - 2p + 1)$.
14. $(m^2 + m + 1)(m^2 - m + 1)(m^4 - m^2 + 1)$.
15. $m^2(m^2 + m + 1)(m^2 - m + 1)$.
16. $(10n^2 + 3n - 1)(10n^2 - 3n - 1)$.

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2. $(2a - b)(4a^2 + 2ab + b^2)$.
13. $(x + 4y)(x^2 - 4xy + 16y^2)$.
14. $(5m + 2n)(25m^2 - 10mn + 4n^2)$.
15. $8(3p - 5q)(9p^2 + 15pq + 25q^2)$.
16. $(1 - 2cd)(1 + 2cd + 4c^2d^2)$.
17. $(k^2 - m)(k^4 + k^2m + m^2)$.

18. $(a^2 + b^2)(a^4 - a^2b^2 + b^4)$. 19. $(3x^2 - 5)(9x^4 + 15x^2 + 25$
20. $(10ab + 1)(100a^2b^2 - 10ab + 1)$.
21. $2(x + 3)(x^2 - 3x + 9)$.
22. $3x(x - 4y)(x^2 + 4xy + 16y^2)$.
23. $d^2(1 + d)(1 - d + d^2)$.
24. $(a - b - c)(a^2 - 2ab + b^2 + ac - bc + c^2)$.
25. $c(c^2 - 9cd + 27d^2)$. 26. $2mn(2m + n)(4m^2 - 2mn + n^2)$
27. $3x(9x^2 + 45x + 75)$. 28. $2y(3x^2 + y^2)$.
29. $(x - y)(x + y)(x^2 + y^2)(x^2 + xy + y^2)(x^2 - xy + y^2)$
$(x^4 - x^2y^2 + y^4)$.
30. $x + y + z$.

Pages 94-95

2. $x - 2$. 3. $x - 3$. 6. $(x - 2)(x^2 + 1$
7. $(x - 1)(x + 1)(x + 7)$. 8. $(x - 1)(x + 2)(3x + 1)$.
9. $(x + 1)(2x^2 - x - 4)$. 10. $(x - 2)(x + 1)(x - 3)$.
11. $(a + 1)(a - 2)(2a + 5)$. 12. $(x + 3)(2x - 1)(2x + 1)$.
13. $(x - y)(x + y)(x - 5y)$. 14. $(a + 5b)(2a^2 - 3b^2)$.
15. $(x - 2)(x + 2)(2x - 1)$. 16. $(x - 6)(x^2 + x + 6)$.
17. $(x + 4)(2x + 1)(2x - 1)$. 18. $a + b + c + d = 0$.
19. $(3x + 4)(x^2 + 2)$. 20. -14. 21. -2.

Exercises on Chapter IV. Pages 97-99

1. $4x(x - 5y)$. 2. $(3a - 5)(a - 3)$. 3. $(n - 10)(n + 10)$.
4. $(m + n)(x + y)$. 5. $(m + 10)(m^2 - 10m + 100)$.
6. $(2x - 3)(2x + 3)$. 7. $(y - 1)(y^2 + y + 1)$.
8. $(p - 2r)^2$. 9. $(2a - 3)(4a^2 + 6a + 9)$.
10. $(1 + 5r)(1 - 5r + 25r^2)$. 11. $(x - 3)(x - 8)$.
12. $m^2n^2(n - m)$. 13. $3(b - 2)(b + 2)$.
14. $(k - 7)(k + 6)$. 15. $(2mn - 1)(2mn + 1)$.
16. $(a - 2b)(x - y)$. 17. $2(r - 2)(r^2 + 2r + 4)$.
18. $(2a + 3)(a - 7)$. 19. $(m + n - k)(m + n + k)$
20. $(r - s - t)(r - s + t)$. 21. $(9x - 7y)(9x + 7y)$.
22. $(x - y)(a + b)$.
23. $(x^2 - xy - y^2)(x^2 + xy - y^2)$.

4. $(a+1)(a-2)(a-3)$.
6. $a(x-1)(x+1)(x^2+1)$.
8. $2(m-2)(m+2)(2m^2-3)$.
9. $(x-y)(x^2-2xy+y^2-3x+3y-1)$.
0. $(2k-n)^2(2k-n-1)$.
2. $3x^2y^3z(2y-5z)(4y^2+10yz+25z^2)$.
3. $(3a^2-7a+2)(3a^2+7a+2)$.
4. $(x+3)(3x^2-5)$.
6. $(11x^2-2xy+10y^2)(11x^2+2xy+10y^2)$.
7. $(2x-3y)(2x+3y+1)$.
8. $(x+2y)(2x-3y-1)$.
0. $(2a+3b)(4a^2-6ab+9b^2-2a+3b)$.
1. $(12x+5y)(5x-8y)$.
3. $2a^2b(12a-b)(a+4b)$.
5. $(r-3s)(r^2+3rs+9s^2+r+3s-1)$.
6. $8x(x+5y-2z)$.
7. $(2-n)(2+n)(2-m)(4+2m+m^2)$.
8. $(x+1)(x-3)(x-1)^2$.
0. $(a^2b+3)^2$.
2. $(x-y+z)^2$.
5. $2x+7$.
8. -3.
0. $(b-a+c)(b+a-c)(a+c-b)(a+c+b)$.
1. (*a*) $(2x-3y)(2x+3y)(4x^2+6xy+9y^2)(4x^2-6xy+9y^2)$.
   (*b*) $(a+b+1)^2$.
2. $(x^2-2xy+2y^2)(x^2+2xy+2y^2)$
   $(x^4+2x^3y+2x^2y^2+4xy^3+4y^4)$
   $(x^4-2x^3y+2x^2y^2-4xy^3+4y^4)$
3. $8a+4b+2c+d=0$.
5. $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.
7. $(x-1)(x+1)(x^2+1)(x^4+1)(x^8+1)(x^{16}+1)(x^{32}+1)$.
9. Either $2ac-b$ or $-2ac-b$ must be a perfect square.

Page 102

1. $x-y$.
4. $3x^3(x-y)$.
12. $a-5$.
15. $2x+3y$.
13. $2(m+4)$.
16. a^2+ab+b^2.

17. $2(3x - 5)$. 18. $x - 2$. 19. $(x + 2)(x + 3)$
20. $c(5c - d)$.

Pages 103-104

1. $\dfrac{a + b}{a^2 + ab + b^2}$. 2. $\dfrac{m + 3}{m + 6}$. 3. $\dfrac{2x - 3y}{x - y}$.

4. $\dfrac{4x}{y(x + y)}$. 5. $\dfrac{z + 1}{z^2 + 1}$. 6. $\dfrac{x + y - z}{x - y - z}$.

7. $\dfrac{x + 5}{x - 3}$. 8. $\dfrac{2x + 3}{x}$. 9. $\dfrac{p^2 - 2pq + 2q^2}{q}$

10. $\dfrac{5a - 3d}{6b + 5c}$. 11. $\dfrac{2c(6c + 7d)}{c + d}$. 12. $\dfrac{m^2 + n^2}{m^2 + mn + n^2}$

Pages 106-107

1. $\dfrac{x + 3}{x - 2}$. 2. $\dfrac{m^2 - mn + n^2}{m^2 + mn + n^2}$. 3. $\dfrac{2xy}{3x - 2y}$.

4. $\dfrac{2y}{21}$. 5. $a + 7$. 6. $\dfrac{m^2 + n^2}{m + n}$.

7. $\dfrac{9p^2 + 3pq + q^2}{9p^2 - 3pq + q^2}$. 8. x/y. 9. 1.

10. abc. 11. $\dfrac{9c^2 - 12cd + 16d^2}{(c + 2d)^2}$. 12. $\dfrac{a^2 + b^2}{(a + b)^2}$.

13. x. 14. $\dfrac{2x^2 + 5}{3x^2 - 4}$. 15. m/k.

16. $\dfrac{x + 2y}{2x + y}$.

Pages 109-110

1. $(x + y)(x - y)^2(x^2 + xy + y^2)$.
2. $(a - 5)(a + 4)(a + 5)(2a + 1)$.
3. $20(m - 4)(m + 4)^2$.
14. $18x^3(x - y)^2$.
5. $(2x - 3y)(2x + 3y)(x - y)(4x^2 - 6xy + 9y^2)$.
6. $a(a - b)(a^2 + ab + b^2)(a^2 - ab + b^2)$.
7. $4(3x - 5)(x + 1)(2x - 1)(3x + 5)$.
8. $(x - 2)(x + 1)(2x + 1)(x^2 + 7)$.
9. $(x - 2)(x + 2)(x - 3)(x + 3)(x - 1)$.
0. $c(5c - d)(5c + d)(2c + 3d)(25c^2 + 5cd + d^2)$.
1. $(x - y)^2(x + y)^2(x^2 + y^2)$.
2. $(x + 1)(x - 3)(x + 7)(2x - 1)$.
3. $(x - y - z)(x + y - z)(x + y + z)^2$.
4. $(7ab + 10c)(7ab - 10c)^2$.
5. $3(2x^2 - 5x - 3)(2x^2 + 5x - 3)$.
6. $(m^2 - 1)(m^4 + m^2 + 1)$.
7. $p(3p + 2q - 5)(3p - 2q + 5)$.
8. $(a + 2)(3a^2 + 4)(2a^2 - 5)$.
9. $4c^2d^2(3c + 5d)(c - 2d)(2c - d)$.

Pages 112-114

1. $\dfrac{x + y}{xy}$.
2. $\dfrac{3c - 5}{bc}$.
3. $\dfrac{13}{10}$.
4. $\dfrac{5z + 7x}{xyz}$.
5. $91/180$.
6. $\dfrac{13za - 7ya + 11yz}{xyza}$.
7. $\dfrac{10a - 13b}{24}$.
8. $\dfrac{m - 5n}{m^2 - n^2}$.
9. $\dfrac{xy}{x^2 - y^2}$.
0. $\dfrac{ab}{(a + b)^2}$.
11. $\dfrac{y}{x^3 - y^3}$.
12. $\dfrac{2p}{(p + q)(p^2 - q^2)}$.
13. $\dfrac{-4rs}{r^2 - s^2}$.
14. $\dfrac{2(x^2 - 5x + 5)}{(x - 2)(x - 4)}$.
15. 0.

16. $\dfrac{2x}{x^2-1}$. 17. $\dfrac{2a+5}{(a+1)(a+2)(a+3)}$.

18. $\dfrac{-a+1}{(a-3)(a-5)}$. 19. 0.

20. $\dfrac{15}{x^2-25}$. 21. $\dfrac{4x^2}{4x^2-1}$.

22. $\dfrac{m-n}{m+n}$. 23. $\dfrac{6x+2}{8x^3+1}$. 24. (*a*) $\dfrac{2a}{a^2-b^2}$;

(*b*) $\dfrac{4a^3}{a^4-b^4}$.

25. $\dfrac{8a^7}{a^8-b^8}$. 26. $\dfrac{-11x^2+19x+4}{(3x-2)(x+3)(2x-3)}$. 27. 1.

28. 0. 29. $\dfrac{8y}{7x+2y}$. 30. $\dfrac{6(a^2-8a+10)}{(a-1)(a+2)(a-3)(a-4)}$

Pages 118-120

4. $6/5$. 5. x. 6. $mn/5$.

7. $2ab/c^3$. 8. $8y^2/x$. 9. mn.

10. $\dfrac{m-3n}{7m-n}$. 11. $\dfrac{x^2+6}{4x-6}$. 12. $\dfrac{21x-14}{25x^2-1}$.

13. $\dfrac{3}{4m-3n}$. 14. $\dfrac{x+y}{x-y}$. 15. $x-y$.

16. $\dfrac{2pq}{p^2+q^2}$. 17. b. 18. $\dfrac{x+y}{4x(x-y)}$.

19. $\dfrac{30(a-1)}{9a^2-4b^2}$. 20. $\dfrac{2(x-5)}{3(x+1)}$. 21. $\dfrac{6q}{p^2+pq+q^2}$.

22. $-x^3/y^3$. 23. n/m.

Review Exercises on Chapter V. Pages 120-124

2. (*a*) 18; (*b*) $2(m - n)$.

3. $\dfrac{a^2(3x - a)}{x(x - a)}$.

4. $\dfrac{x - 3}{x^2 + 9}$.

5. $\dfrac{(m + 5)^2}{2m(m - 5)}$.

7. (*a*) 1575; (*b*) $3(x^3 + 8)$.

8. $\dfrac{2}{a - 3}$.

9. $\dfrac{23}{5k - 4}$.

10. $\dfrac{-3(2a - 1)}{a(a^2 - 1)}$.

11. $\dfrac{y^2}{x^3 - y^3}$.

12. $\dfrac{3m - 5}{(m - 4)(m + 1)(m - 3)}$.

13. $\dfrac{m^2 - mn + n^2}{m^2 + mn + n^2}$.

14. 1.

15. $\dfrac{z + 2}{z(1 - z^2)}$.

16. $\dfrac{1}{x + y}$.

17. $\dfrac{1}{2c - 1}$.

18. $\dfrac{8}{n^8 - 1}$.

19. 0.

20. $1/(x^2 - 1)$.

21. $\dfrac{1}{2k^2 - 1}$.

22. 0.

23. $\dfrac{1}{m^2 - mn + n^2}$.

24. 0.

25. $\dfrac{r^2 - rs + s^2}{(r - s)(r^4 - s^4)}$.

26. $\frac{1}{2}$.

27. $\dfrac{15}{(x - 1)(x - 2)(x - 3)}$.

28. 1.

29. $-\dfrac{4c + 2}{7c + 11}$.

30. $\dfrac{x + 3}{x - 3}$.

31. $\dfrac{a^3 - 1}{a^2 + a + 1}$.

32. $-\dfrac{2x^2 + 5}{5}$.

33. $2 - x$.

34. a.

35. $a + b$.

37. y.

38. $x = \dfrac{b(1 - b^2)}{1 - 3b^2 + b^4}$.

39. $\dfrac{625000}{x^8 - 390625}$.

40. $\dfrac{15}{a^2-25}$. **41.** (*a*) $\dfrac{pq}{p+q}$; (*b*) $\dfrac{qf}{q-f}$. **42.** $\dfrac{1-2m}{1-2m+2m^2}$.

43. $\dfrac{-1}{m(m+n)}$. **44.** x/y. **45.** $\dfrac{(2x-3y)(2y-x)}{(3x-2y)(2x-y)}$.

Pages 130-131

28. a^{2x}. **29.** m^{2b}. **30.** a^{2y}.

31. $p^4q^6r^8$. **32.** x^{4m}. **33.** a^{2x2}.

34. $5a^{5-x}b^{x-4}$. **35.** 1. **36.** 1.

37. 2^{n^2}. **38.** 3^{3a-4}. **39.** yz/x^2.

40. a^{2x-2y}. **41.** 1. **42.** (*a*) 2^{12}; (*c*) 5^9.

43. (*a*) 125^4; (*c*) $(a^r)^8$. **44.** (*a*) 5; (*b*) 1. **45.** 81.

46. 2^n.

Pages 137-138

43. 3. **44.** 1/3. **45.** 125.

46. 1/125. **47.** 1. **48.** $\frac{1}{4}$.

49. 0.1. **50.** 1. **51.** 1/8.

52. 2. **53.** 0.027. **54.** 64/27.

55. 1/25. **56.** 3. **57.** 1.

58. 3. **59.** $\frac{1}{2}$. **60.** 32.

61. 1. **62.** 8. **63.** 512.

64. 25. **65.** 0.01. **66.** 243.

67. 1/5. **68.** 6. **69.** 6/5.

70. 11/10. **71.** 3/2. **72.** 125.

73. $\frac{1}{2}$. **74.** 17.5. **75.** 2744.

76. 8. **77.** $2x^2z^3/3y$. **78.** $\dfrac{9y^2}{4x^2a^4}$.

79. $2^{7/12}$. **80.** $\sqrt{b}$. **81.** $16\frac{2}{3}$.

82. 11.

Review Exercises on Chapter VI. Pages 140-143

1. (*a*) $2x^{10}$; (*b*) $6a^{\frac{3}{2}}$; (*c*) -7.
2. (*a*) $3k^3$; (*b*) $3x^{1/5}$; (*c*) $1.5a$.
3. (*a*) y^4; (*b*) $x^{-\frac{3}{2}}$; (*c*) $a^{\frac{3}{2}}$.
4. (*a*) $2x^{-1}$; (*b*) $3a^{\frac{2}{3}}$; (*c*) $5m^{-\frac{3}{8}}$.
5. (*a*) 4; (*b*) $\frac{1}{4}$; (*c*) 4; (*d*) 1/8.
6. 2.
7. $6x + 2x^{\frac{1}{2}} - 20$.
8. $25x^{4/3} - 9$.
9. $2x^{1/3} - 1$.
10. $a^{-2} + 3a^2$.
11. (*a*) $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y$; (*b*) $a^{\frac{2}{3}} - 2a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.
12. x.
13. $\dfrac{2x^{\frac{1}{2}} - 3}{3x^{\frac{1}{2}} - 4}$.
14. $6m^{\frac{2}{3}} + m^{\frac{1}{3}} - 15 - 4m^{-\frac{1}{3}} + 6m^{-\frac{2}{3}}$.
15. $6\frac{3}{4}$.
16. $7x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1$.
17. $c^{5/4} + 4c - 11c^{1/2} - 6c^{1/4}$.
18. $5x^{\frac{1}{4}} - 3y^{\frac{1}{4}}$.
19. $2x - 3x^{\frac{1}{2}} - 2$.
20. $a^{4/3} - 4a^{2/3} + 3 - 6a^{-1/3}$.
21. (*a*) $\sqrt[3]{2}$; (*b*) $\frac{1}{2}$.
22. $2x^{\frac{1}{3}} - y^{\frac{1}{3}}$.
23. $x^{3a} + x^{2a}y^a + x^a y^{2a} + y^{3a}$.
24. $3a^{1/5} + a^{-1/5} - 6a^{-3/5}$.
25. $16a^{-3} + 6a^{-2} + 5a^{-1} - 6$.
26. $a^{3/2}b^{-1} + 2a^{3/4}b^{-1/2} + 1 - a^{-1/2}b^{1/2}$.
27. $9x^3y + 12x^{\frac{3}{2}}y^{\frac{1}{2}} - 2 - 4x^{-\frac{3}{2}}y^{-\frac{1}{2}} + x^{-3}y^{-1}$.
28. $x^{\frac{3}{2}}yz^2$.
29. 0.
30. $m^{\frac{2}{3}} + m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}}$.
31. 3^{n^2}.
32. $8x^{-2} - 7x^{-1} + 6$.
33. $x^{\frac{1}{3}} - 2$.
34. $9/4$.
35. 245.
36. $\frac{1}{2}$.
37. 80.
38. $5a + 8\sqrt{a} - 4$.
39. (*a*) 9; (*b*) 8; (*c*) 4.
40. $9x^{4a} - x^{2a} + 10x^a - 25$.
41. $9a^3b - 30a^{5/2}b^{3/2} + 13a^2b^2 + 20a^{3/2}b^{5/2} + 4ab^3$.
42. $15x - 3x^{\frac{1}{3}} - 2x^{-\frac{1}{3}} + 10x^{-1}$.
43. $x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 1 + 2x^{-\frac{1}{2}} + x^{-\frac{2}{3}}$.

44. $x^{\frac{1}{2}} - 2x^{1/6} + 4x^{-1/6} - 8x^{-\frac{1}{2}}$. 45. $2a^{\frac{1}{4}} - 3a^{-1/12} - a^{-5/12}$.
46. $4x^4 - 6x^{8/3} - 4x^{4/3} - 3 + 10x^{-4/3} + 3x^{-8/3}$.
47. $\frac{1}{2}(e^{2x} + e^{-2x})$. 48. $a + b + c - 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$.
49. $4m^2 + 3m + 2 - 3m^{-1}$. 50. 2.

Page 145

1. 22/7. 2. 18/5. 3. 5.
4. 3/2. 5. 2. 6. 2.5.
7. 18.2. 8. 20. 9. 32.
10. 13. 11. -5. 12. 11.
13. -3. 14. $-1/3$. 15. -5.

Pages 147-148

1. $\frac{c-b}{a}$. 2. $\frac{3a-7b}{5}$.
3. $\frac{k-n}{m-h}$. 4. $\frac{ra-sb}{r-s}$.
5. 1. 6. $p - 1$.
7. $2m/n$ 8. $a - 2b$.
9. $-ab$. 10. cd.
11. p/q. 12. 1.
13. $\frac{ab+bc+ca}{a+b+c}$. 14. $a + b - 1$.
15. c. 16. $\frac{2ab}{a^2-b^2}$.
17. $E = CR$, $R = E/C$. 18. $IC/100$. 19. lr^{1-n}.
20. $n = \frac{2s}{a+l}$, $a = \frac{2s}{n} - l$, $l = \frac{2s}{n} - a$. 21. $2E/v^2$.
22. $r = \frac{s-a}{s-l}$, $l = \frac{rs+a-s}{r}$, $a = s - rs + rl$,

23. $h = \dfrac{6V}{B_1 + B_2 + 4M}$, $B_1 = \dfrac{6V}{h} - B_2 - 4M$,
$M = \dfrac{3V}{2h} - \dfrac{B_1 + B_2}{4}$.

24. $u = s/t - \frac{1}{2}at$, $a = 2(s - ut)/t^2$,

25. $a = (v^2 - u^2)/2s$, $s = (v^2 - u^2)/2a$.

Pages 149-151

1. 8.
2. 12.
3. 23/9.
4. 2/5.
5. 1/7.
6. 29/4.
7. $-1/6$.
8. $\dfrac{2ab}{a + b}$.
9. $\dfrac{b - d}{c - a}$.
10. $\dfrac{a + b}{a - b}$.
11. $\dfrac{bm + an}{n - m}$.
12. $-4/9$.
13. 3.
14. 2.
15. -3.
16. 0.
17. 5.
18. $-11/7$.
19. $-5/3$.
20. 9.
21. -2.
22. 1/7.
23. -5.
24. 1/6.
25. $\dfrac{2ab - ac - bc}{a + b - c}$.

Page 152

1. 4, -9.
2. 2, -4.
3. $-2/5$, $-5/2$.
4. 1/3, $-10/11$.
5. -1, -2, -3.
6. 1/2, 2/3, 3/4.
7. $-a$, b.
8. b/a, $-d/c$.
9. 2/3, $-4/5$.
10. 10, 15.
11. 1.8, 1.5.
12. p, $q/2$, $-r$, $-s/2$.

Pages 153, 155

1. ± 3.
2. ± 2.
3. $\pm 9/5$.
4. -7.
5. 2, 3.
6. 3, -8.
7. 3/2, -2.
8. 3, $-4/3$.
9. 4, 2/7.

10. $-5/3$. **11.** $\pm 9/4$. **12.** $10/9, 9/10$.
13. $5a/6, -3a/2$. **14.** $-4r/5, -9r/4$. **15.** $5/k, -3/2k$.
16. $3, k$.

Pages 156-157

2. $\pm 1.2, \pm 0.9, \pm 6/5, \pm 13/11, \pm 9/2$.
3. ± 15. **4.** $\pm 3/2$. **5.** $2/5$.
6. $7, -5$. **7.** $0, -20$. **8.** $7, -1/3$.
9. $1, -5$. **10.** $19/8, 5/8$. **11.** $-a \pm b$.
12. $(q - b)/(a - p), -(b + q)/(a + p)$.
13. $-3m - 2n, (2n - 3m)/5$.
14. $\pm 1, \pm 3$. **15.** $\pm 3, \pm 1/3$. **16.** $\pm a, \pm 5a$.

Pages 157-158

1. 7. **2.** 6. **3.** 5/6.
4. -24. **5.** 4. **6.** 11/21.
7. 7/20. **8.** 5/14. **9.** 4 or -4.
10. 5/6 or 6/5. **11.** 8, 14, or $-8, -14$. **12.** 3 or -1.
13. 2 or -9. **14.** 9/15. **15.** 7/15.

Pages 159-161

5. $25b$. **6.** $100b/a$. **7.** 36, 24.
8. $\$6x$, $\$3y$. **9.** $\$a(1 + b/100)$, $\$100a/(100 - b)$.
10. $\dfrac{b - a}{a}\ 100\%$. **11.** \$120. **12.** \$50.
13. 25 lb. **14.** $6\frac{2}{3}$ lb. **15.** 150 lb.
16. \$20. **17.** $bc/(a + b), ac/(a + b)$.
18. $44\%, 56\%$. **19.** \$750, \$250. **20.** $\$\dfrac{c - b}{a - b}\ 100$.

Pages 162-164

1. 20 rods, 24 rods. **2.** 8 in. **3.** 30 ft. by 40 ft.
4. 9 in., 16 in. **5.** $6\frac{2}{3}$ ft. **6.** 24 in.

7. 28 ft. by 56 ft. 8. $12\sqrt{2}$ ft.
9. A: 40 rods by 40 rods; B: 32 rods by 50 rods.
10. $4\frac{4}{5}$. 11. $\frac{1}{2}$ in., $\frac{3}{4}$ in. 12. 2 in.
13. $2\frac{1}{2}$ in., 5 in. 14. 4 ft. 15. 4 in.

Pages 165-166

13. 36 m.p.h. 14. 36 m.p.h., 40 m.p.h.
15. 6 m.p.h. 16. 10.48 A.M., 36 miles.
17. $3\frac{1}{2}$ m.p.h., 3 m.p.h. 18. 6 m.p.h., 8 m.p.h.

Exercises on Chapter VII. Pages 167-173

1. $7\frac{1}{2}$. 2. 1/3. 3. 12.
4. 15. 5. -7. 6. 1.
7. $(a+b)/(a-b)$. 8. a. 9. $(pr-qs)/(ps-qr)$.
10. 0, 7. 11. 0, ± 3. 12. ± 6.
13. 3. 14. -2. 15. 6.
16. 1/3. 17. -2. 18. 3/7, $-2/5$.
19. $-4/3$, $5/2$, $-9/4$. 20. 6, -5. 21. 3, -5.
22. 11, -12. 23. 1, 1/10. 24. 4, $-3/2$.
25. $3/2$, $-2/3$. 26. 10, 11, 12. 27. 15, 20.
28. 17/25. 29. 1/2. 30. 3.
31. 75, 50. 32. \$180, \$120. 33. 3800.
34. 20 rods by 30 rods. 35. 18 sq. units. 36. 4 ft.
37. 35c. 38. \$75, \$125. 39. 160 miles.
40. 4 m.p.h., 3 m.p.h. 41. 36. 42. 73.
43. $11\frac{1}{9}$ lb. 44. $200m$, $r/200$ min., 1200 deg. per sec.
45. 12. 46. 8. 47. 7.
48. 7. 49. -1. 50. -4.
51. $7\frac{1}{2}$. 52. $-1/2$. 53. -2.
54. 4.8. 55. 1.5. 56. 3, 1/3.
57. 2, 3. 58. -4, $-1/2$. 59. 11/3, 13/2.
60. ± 2, ± 3. 61. $\pm a$, $\pm b$, $\pm c$. 62. 7, 4.

63. (*a*) -1, (*b*) $1/2$. 64. (*a*) Numbers between -1 and 1; (*b*) Numbers greater than 1 or less than -1

65. $\dfrac{V}{\pi(r_2^2 - r_1^2)}$. 66. $24\pi a$. 67. Width, $ab/(a - 2b)$

68. 100 miles from the town; 5 P.M. 69. $2(ab + cd)/h$ in.

70. $\dfrac{5(24h + 5)}{12(12h - 5)}$ ft. 71. \$95. 72. 62.

73. 6 in., 15 in. 74. 6 in. 75. Decreased by 1 sq. in

76. 32 sq. in. 77. 703. 78. 494.

79. Yes, 5 ranks. 80. 5 m.p.h.

81. 96 miles from A; 11.24 A.M. 82. 100, 40.

85. (*a*) 12, (*b*) $5/3$. 86. No. 87. $p = m + n$, $q = mn$.

88. ± 2, ± 3. 89. -2, ± 3, 4. 90. 1, -2, 3, -4.

91. 1, k/h. 92. 1, $(c - a)/(a - b)$. 93. 6, 8.

Page 178

1. $4a + 7$. 2. $6b - 5$. 3. $8x - 3y$.

4. $\frac{2}{3}x^2 - \frac{1}{2}y$. 5. $\dfrac{p}{3} - \dfrac{q}{2}$. 6. $\frac{1}{2}ab + 1$.

7. $10m + 5n$. 8. $x^{-1} - 9$. 9. $3x - 4y + 5z$.

10. $10m^2 - 4n^2 + 6$. 11. $x^2 + x^{-2}$. 12. $\frac{2}{5}c + \frac{3}{4}d - \frac{5}{2}$.

13. $a^{\frac{1}{3}} + 6b^{\frac{1}{3}}$. 14. $7a^{\frac{1}{2}} + 2b^{\frac{1}{2}} - 3c^{\frac{1}{2}}$. 15. $a + b - \frac{1}{2}$.

16. $3a^2b - 5b^2c - 7c^2a$. 17. $x^{\frac{1}{5}}y^{\frac{2}{5}} - \sqrt{5}$.

18. $9x^{\frac{1}{3}} - 8y^{\frac{1}{4}} + 6z^{\frac{1}{5}}$. 19. $3(x^2 + 1) - 4y$.

20. $2a + 1 - 2^{-1}b^{-1}$. 21. $ab^{-1} - 2bc^{-1} + 5ca^{-1}$.

22. $x^4 + x^3 + x$.

Pages 181-182

1. $3x^2 - 5x + 2$. 2. $8x^2 + 3x - 6$.

3. $7c^3 - 4c^2 + 3c$. 4. $5a^2 - 7ab + 4b^2$.

5. $6a^2b^2 - 5ab - 4$. 6. $9m^4 + 10m^2 - 11$.

7. $2a^3 - a^2 + 3a - 4$.
8. $x^3 + 2x^2 - 5$.
9. $2y^3 - 3y + 7$.
10. $2p^3 - 5p^2 + 6$.
11. $3k^3 - 4k^2 + 5k - 2$.
12. $10p + 9 + 10p^{-1}$.
13. $a^{3/2} + 2a^{1/2} - a^{-1/2} - 2a^{-3/2}$.
14. $x^{2/3} - 4x^{1/3} + 5$.
15. $a^{-2/5} + 6a^{-1/5} - 4$.
16. $4a^{-3} - 6a^{-1} + 9a$.
17. $11n^{-1} - 8n^{-2} + 7n^{-3} - 9n^{-4}$.
18. $5x^{\frac{1}{2}} - 6 + 7x^{-\frac{1}{2}}$.
19. $4x^{1/3} - 7x^{1/6} - 5$.
20. $3y^{-3/5} - 4y^{-2/5} + 5y^{-1/5}$.
21. $x^3 + 3x + 2$.
22. $2y^{3/2} - 3y^{1/2} + 6y^{-1/2} - 4y^{-3/2}$.
23. $2x^{-3} - 5x^{-2} + 3x^{-1} - 4$.
24. $x^4 + 2x^3 - 3x^2 + 4x - 5$.
25. $3x^{1/2} - 5x^{1/4} - 2 - 4x^{-1/4} + 6x^{-1/2}$.

Page 184

1. 2, 2, 2, 3, 3, 3.
2. 47.
3. 79.
4. 88.
5. 97.
6. 234.
7. 256.
8. 31.4
9. 318.
10. 36.7.
11. 528.
12. 5.95.
13. 572.
14. 6.64.
15. 823.
16. 0.951.
17. 9.21.
18. 25.63.
19. 3726.
20. 54.18.
21. 683.7.
22. 72.56.

Page 186

1. 16.2.
2. 2.53.
3. 2.89.
4. 1.414.
5. 1.732.
6. 2.236.
7. 27.33.
8. 2.733.
9. 8.643.
10. 7.649.
11. 0.7649.
12. 2.419.
13. 1.811.
14. 18.11.
15. 5.727.
16. 8.417.
17. 2.329.
18. 15.39.

Exercises on Chapter VIII. Pages 186-188

1. (*a*) 4, (*b*) 6, (*c*) 8.
2. (*a*) 2, (*b*) 8.
3. (*a*) $7x - 3a$, (*b*) $9x - 4y$, (*c*) $3c + 5c^{-1}$, (*d*) $2p^{-2} + 4p^{-4}$.
4. (*a*) $a - 2b^2 + 3c^2$, (*b*) $3x + 5y + 7z$, (*c*) $2p + 5q - 10r$.

5. $x^3 - 4x^2 + 2x$.
6. $3y^2 - 2y - 7$.
7. $9p^4 + 7p^2 - 5$.
8. $5x^3 - 3x + 4$.
9. $x^3 + x^2 + x + 1$.
10. $7a^5 - 5a^3 + 4a^2$.
11. 888.
12. 27.7.
13. 53.0 cm.
14. 25.4 in.
15. 19.4 in., 38.9 in.
16. 19.4 in.
17. No.
19. 6828.
20. 7.391.
21. 23.37.
22. $2z^3 - 3z^2 + z - 5$.
23. $5a^{2/3} - 2a^{1/3}b^{1/3} - 6b^{2/3}$.
24. $7x^{-3}y^{-1} - 4x^{-2}y^{-2} + 3x^{-1}y^{-3}$.
25. $10h^{1/2} - 6h^{1/4}\,k^{1/4} + 5k^{1/2}$.
26. $2x^{-1/3} + 5x^{-2/3} + 7x^{-1}$.
27. 19.8 in.
28. 10.82 in.
29. 7.00 in.
30. 31 rods.
31. 5.2 in.

Page 191

1. 24/1.
2. 42/5.
3. $-61/10$.
4. 65/4.
5. $-17/3$.
6. 212/99.
7. 21/10.
8. $-4/45$.
9. 145/44.
10. $-7/360$.

Page 193

The irrational numbers are those of Exercises 3, 6, 8, 10, 14, and 16.

Pages 197-198

1. $\sqrt{35}$.
2. $\sqrt{320}$.
3. $\sqrt{24}$.
4. $\sqrt{abxy}$.
5. $\sqrt{30ab}$.
6. $\sqrt{132xy}$.
7. $\sqrt{12}$.
8. $\sqrt{63}$.
9. $\sqrt{75}$.
10. $\sqrt{99}$.
11. $\sqrt{128}$.
12. $\sqrt{60{,}000}$.
13. $\sqrt{5a^2}$.
14. $\sqrt{25a}$.
15. $\sqrt{a^2b}$.
16. $4\sqrt{2}$
17. $5\sqrt{2}$.
18. $4\sqrt{5}$.
19. $3\sqrt{5}$.
20. $2\sqrt{6}$.
21. $10\sqrt{7}$.
22. $9\sqrt{2}$.
23. $5\sqrt{5}$.
24. $6\sqrt{2}$.
25. $2a\sqrt{b}$.
26. $2m\sqrt{2m}$.
27. $3r^2\sqrt{2r}$.

28. $ab\sqrt{ac}$. 29. $x^2y^2\sqrt{y}$. 30. $5x^3y^4\sqrt{xy}$.
31. $\sqrt{238}$. 32. $\sqrt{180}$. 33. $\sqrt{1728}$.
34. $\sqrt{45a^2b^4}$. 35. $\sqrt{12xy^2}$. 36. $\sqrt{4500x^2y}$.
37. $6\sqrt{7}$. 38. $5\sqrt{11}$. 39. $13\sqrt{2}$.
40. $ab^2c^3\sqrt{abc}$. 41. $7xy\sqrt{2y}$. 42. $9n^2\sqrt{7m}$.
43. $6\sqrt{3}$ cm. = 10.4 cm. 44. 14.0 cm. 45. $4\sqrt{41}$ ft. = 12.8 ft.
46. $18\sqrt{5}$ in. = 40.2 in. 47. $64\sqrt{21}$ sq. in = 293 sq. in.
48. $3a\sqrt{2}$ in. 49. $b^2\sqrt{3}$ sq. in. 50. $2s\sqrt{5}$ ft.
51. $3k^2\sqrt{15}$ sq. cm.

Page 199

1. $2\sqrt{3}$. 2. $\sqrt{7}$. 3. $\sqrt{2}$.
4. $3\sqrt{2}$. 5. $\sqrt{5}$. 6. $5\sqrt{5}$.
7. $3\sqrt{3}$. 8. $4\sqrt{2}$. 9. $6\sqrt{10}$.
10. $3\sqrt{2}$. 11. $3\sqrt{7}$. 12. $4\sqrt{2}$.
13. $a\sqrt{3}$. 14. $-2\sqrt{11}$.

Pages 201-202

1. $40\sqrt{42}$. 2. 36. 3. $120\sqrt{2}$.
4. $17 + 7\sqrt{5}$. 5. $27 - \sqrt{3}$. 6. $\sqrt{2}$.
7. 10. 8. 2. 9. 3.
10. 9. 11. 6. 12. 1.
13. $8 + 2\sqrt{15}$. 14. $13 - 4\sqrt{3}$. 15. $6 + 2(\sqrt{2} + \sqrt{3} + \sqrt{6})$.
16. $25\sqrt{10}$. 17. 36. 18. 100.
19. $84\sqrt{30}$. 20. 24. 21. $23 + 4\sqrt{33}$.
22. $73 - 12\sqrt{35}$. 23. $94 + 24\sqrt{15}$. 24. $74 - 26\sqrt{15}$.
25. $3(35\sqrt{3} - 12\sqrt{5})$. 26. 2. 27. 2.
28. 5. 29. $5 + 4\sqrt{2} + 3\sqrt{3} + 2\sqrt{6}$.
30. $a + b + c + 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$.
32. $39 + 12\sqrt{3}$ sq. in. = 59.8 sq. in.
33. $34 + 30\sqrt{2}$ sq. in. = 76.4 sq. in.
34. $\frac{5}{2}\sqrt{5} + 2\sqrt{10}$ sq. in. = 11.9 sq. in.
35. $44(4 - \sqrt{7})$ sq. cm. = 59.6 sq. cm. 36. 14 sq. in.
37. $\frac{88}{7}(8\sqrt{3} + 9)$ sq. in. = 287 sq. in.

Pages 205-207

1. $\sqrt{17}$.
2. $\sqrt{6}$.
3. $\sqrt{19}$.
4. $\sqrt{11}$.
5. $\frac{1}{2}\sqrt{42}$.
6. $\frac{1}{3}\sqrt{75}$.
7. $\sqrt{5} + 2$.
8. $\sqrt{10} - 3$.
9. $4 + \sqrt{15}$.
10. $\frac{1}{3}\sqrt{42}$.
11. $\frac{5}{2}\sqrt{3}$.
12. $\frac{1}{5}\sqrt{10}$.
13. $\frac{5}{3}(4 + \sqrt{7})$.
14. $\frac{2}{31}(6 + \sqrt{5})$.
15. $\sqrt{30} - \sqrt{22}$.
16. $10\sqrt{3} + 12\sqrt{2} = 34.3$.
17. $(39 + 10\sqrt{14})/11 = 6.95$.
18. $13 + 4\sqrt{11} = 26.3$.
19. $24 - 9\sqrt{7} = 0.186$.
20. $(76 - 15\sqrt{15})/49 = 0.365$.
21. $(100 - 29\sqrt{10})/53 = 0.157$
22. $1/(4 + \sqrt{15})$.
23. $1/(2 + \sqrt{3})$.
24. $1/(\sqrt{102} + 10)$.
27. 12, Greater.
28. 14, Greater.
29. The second.
30. The first.
31. $\frac{1}{2}\sqrt{6}$.
32. $5(\sqrt{2} + \sqrt{3})$.
33. $\sqrt{5} - \sqrt{3}$.
34. 1.
35. $9 + 4\sqrt{5}$.
36. $(8 + 3\sqrt{6})/10$.
37. $x = 5\sqrt{2}/2, y = \sqrt{3}/3$.
38. $x = 3\sqrt{2} + 2,\ y = 2\sqrt{2} - 3$.
39. $x = 3\sqrt{3} - 2\sqrt{2}, y = 2\sqrt{3} - 3\sqrt{2}$.
40. $x = 6\sqrt{5} - 5\sqrt{3}, y = 4\sqrt{15} - 15$.

Exercises on Chapter IX. Pages 207-211

3. $\sqrt{36} = 6$, $\sqrt{0.36} = 0.6$, $\sqrt{625} = 25$, $\sqrt{6.25} = 2.5$, $\sqrt{0.0625} = 0.25$; Others irrational.
4. $\sqrt{245}$, $\sqrt{108}$, $\sqrt{\pi r^2}$, $\sqrt{a^2b^4c}$.
5. $4\sqrt{11}$, $5\sqrt{13}$, $2a\sqrt{2b}$, $2bc\sqrt{3ac}$.
6. 0.
7. $2\sqrt{7}$.
8. (*a*) $6\sqrt{6}$, (*b*) 40, (*c*) $14\sqrt{30}$, (*d*) $20\sqrt{10}$, (*e*) $16\sqrt{6}$.
9. (*a*) $\sqrt{6}$, (*b*) 2, (*c*) 1, (*d*) $\sqrt{3}$, (*e*) 2/3.
10. (*a*) 30, (*b*) 210, (*c*) 273.
11. (*a*) -1, (*b*) 3, (*c*) 6.
12. $\sqrt{21} - \sqrt{17}$, 4.
13. $9 + 4\sqrt{5}$.
14. (*a*) $3(\sqrt{7} + 2)$, (*b*) $2(\sqrt{10} + \sqrt{6})$.
15. $2\sqrt{3}$.
16. (*a*) $7 + 4\sqrt{3}$, (*b*) $11 - 6\sqrt{2}$, (*c*) $21 + 8\sqrt{5}$
17. (*a*) $3 + 2\sqrt{5}$, (*b*) $1 + \sqrt{7}$, (*c*) $9 - \sqrt{11}$.
18. (*a*) $4\sqrt{2}$, (*b*) 81, (*c*) $4\sqrt[3]{4}$, (*d*) 4, (*e*) $\sqrt{5}$.
19. (*a*) $2\sqrt[3]{2}$, (*b*) $3\sqrt[3]{2}$, (*c*) $10\sqrt{3}$, (*d*) $2\sqrt{3}$, (*e*) $5\sqrt{4}$.

20. (*a*) $\frac{1}{3}\sqrt{2}$, (*b*) $\sqrt{2}$, (*c*) 3.

21. (*a*) $a^2 - b$, (*b*) $p^2 - q^2r$, (*c*) $a - x^3$.

22. (*a*) b, (*b*) $3b$, (*c*) $2b$.

23. (*a*) $29 + 12\sqrt{5}$, (*b*) $43 - 30\sqrt{2}$, (*c*) $14 - 8\sqrt{3}$.

24. (*a*) 12, (*b*) 40, (*c*) 10. **25.** 5/11. **26.** 14.

27. $\sqrt{3} - \sqrt{2}$. **28.** $(2\sqrt{7} + 1)/27$. **29.** 12.

30. $(3 - \sqrt{2})(4 - \sqrt{5})(3 + \sqrt{2})(4 + \sqrt{5})$.

31. $10 + 2(\sqrt{6} + \sqrt{10} + \sqrt{15})$. **32.** -8.

33. $(\sqrt{5} + \sqrt{3} + \sqrt{2})(\sqrt{5} + \sqrt{3} - \sqrt{2})(\sqrt{5} - \sqrt{3} + \sqrt{2})$ $(-\sqrt{5} + \sqrt{3} + \sqrt{2}) = 24$.

34. $17 + 12\sqrt{2}$. **35.** $36\sqrt{7} + 19\sqrt{70} - 32\sqrt{10} - 144$.

36. (*a*) $7\sqrt{6} + 15$, (*b*) $\sqrt{35} + 2$. **37.** $\sqrt{30} - 2$.

38. $13\sqrt{2} - 9\sqrt{3} = 2.79$.

39. (*a*) $(\sqrt{5} - 1)(\sqrt{6} + 2)$, (*b*) $(\sqrt{5} + \sqrt{7})(\sqrt{2} + \sqrt{3})$.

40. (*a*) 2, (*b*) 0. **41.** 14/3.

42. In the order as written. **43.** $\sqrt{a} + \sqrt{b}$.

44. $22\sqrt{10} = 70$, app. **45.** $4\sqrt{3}$ ft. = 6.9 ft.

46. $\frac{5}{2}\sqrt{2}$ sec. = 3.5 sec., $\sqrt{5}$ sec. = 2.4 sec.

47. (*a*) 2 sec., (*b*) $2\sqrt{2} - 2$ sec. = 0.83 sec., (*c*) $2\sqrt{3} - 2\sqrt{2}$ sec. = 0.64 sec.

48. 2.5 acres. **49.** $8\sqrt[4]{12}$ in.

50. $4\sqrt{10}$ cm. **51.** (*a*) $6\sqrt{5}$ ft., (*b*) $12\sqrt{5}$ ft.

52. $\frac{15}{4}\sqrt{7}$ sq. in. **53.** (*a*) $3\sqrt{41}$ ft. = 19.2 ft., (*b*) $15\sqrt{2}$ ft. = 21.2 ft.

54. $4\sqrt{14}$ sq. in. = 15.0 sq. in.

56. (*a*) $x = y = z = 0$; (*b*) 1, 0, 0, in any order; (*c*) 1, 1, 0, in any order; 4, 0, 0, in any order; (*d*) 1, 1, 1; 4, 1, 0, in any order; 9, 0, 0, in any order; (*e*) 16, 0, 0, in any order; 9, 1, 0, in any order; 4, 4, 0, in any order; 4, 1, 1, in any order.

57. $(7 + 2\sqrt{15} - \sqrt{5} - 3\sqrt{3})/11$.

58. $(2\sqrt{3} + 3\sqrt{2} - \sqrt{30})/12$.

59. $\sqrt{7} + 9\sqrt{3} + 5\sqrt{5} - 2\sqrt{105}$.

60. $\sqrt{3} + 1$, $\sqrt{5} + 3$, $\sqrt{6} + 5$.

61. $\sqrt{6} + \sqrt{5} - \sqrt{3}$. **62.** $\sqrt{5} + \sqrt{2}$.

Page 213

1. 4.
2. 36.
3. 100.
4. 144.
5. 9.
6. 49.
7. 81.
8. 225.
9. b^2.
10. $16a^2$.
11. $49b^2$.
12. $9c^2$.
13. 1/4.
14. 25/4.
15. $81a^4/4$.
16. $121b^4/4$.
17. $q^2/4$.

Page 217

1. 3, −5.
2. 7, 3.
3. −5, −7.
4. 5/2, −3.
5. 7/3, −3/2.
6. 10, −9.
7. $1 \pm \sqrt{22}$.
8. $-2 \pm \sqrt{3}$.
9. $7 \pm 2\sqrt{6}$.
10. 5, −27.
11. 1/5, −5/2.
12. $-3 \pm \sqrt{7}$.
13. $(-11 \pm \sqrt{13})/2$.
14. $(-2 \pm \sqrt{14})/5$.
15. $(11 \pm \sqrt{13})/18$.
16. −2/7, −7/2.
17. 0.2, −1.2.
18. −0.9, −4.1.
19. 3.4, 1.9.
20. 3.8, 0.8.

Page 218

1. 2, 7, −4.
2. 6, −8, 3.
3. 1, −7, −12.
4. 1, −2, 5.
5. 5, −5, −24.
6. 2, 2, 1.
7. 1, 14, 15.
8. $1, -p, -q$.
9. $1, -2a - b, a^2 - c$.
10. $m^2 - p^2, 2(mn + pq), n^2 - q^2$.
11. $r^2, 0, 1 + 2r$.
12. $k^2, 3hk, 0$.

Pages 220-221

2. 7, 2.
3. 11, 7.
4. 5, −6.
5. 3/2, −4.
6. 7/2, −5/2.
7. $5a, 9a/2$.
8. $b/3, -b/5$.
9. $(-3 \pm \sqrt{5})/2$.
10. $2 \pm \sqrt{6}$.
11. $4 \pm \sqrt{10}$.
12. $-5 \pm 2\sqrt{3}$.
13. $(-5 \pm \sqrt{13})/2$.
14. $(9 \pm \sqrt{33})/12$.
15. 5/12, −7/6.
16. $(2 \pm \sqrt{11})/3$.
17. 27, −43.
18. $(11 \pm 5\sqrt{2})/3$.
19. −17/2, −26/3.
20. 25/6, −18/5.
21. $(7 \pm \sqrt{15})/8$.

22. $(-p \pm \sqrt{p^2 - 4q})/2$.
23. $m \pm \sqrt{m^2 - n}$.
24. $(-2b \pm \sqrt{4b^2 + 9ac})/a$.
25. $(-R \pm \sqrt{R^2 - 4L/C})/2L$.
26. $(-7 \pm \sqrt{85})/6k$.
27. $(20 \pm 8\sqrt{10})/3$.
28. $(3 \pm \sqrt{9 + 12ac})/2c$.
29. $(-5 \pm 2\sqrt{10})/3$.
30. $(1 \pm \sqrt{5})/2$.
31. $(7 \pm \sqrt{609})/28$.

Pages 225-226

2. 4, −2.
3. 0, 2.
4. 1, 1.
5. 2.4, −0.4.
6. No real roots.
8. 0, 3.
9. 1, 2.
10. 4, −1.
11. 4.4, −1.4.
14. 8, 1.
15. 6.5, 0.5.
17. 2, 2.
18. 0, 4.
19. 4.8, −0.8.
21. No; $x = -2$; least value = 1.
22. −1, −3.
23. −2, −2.
24. 1, −5.
26. 1, −2.
27. 3, −4.
28. 9.
29. $-x^2 - 5x$.
30. A peak.
33. 3, 5.
34. 4, 4.

Pages 228-231

1. 11 and 9 or −9 and −11.
2. 5 and 6 or −6 and −5.
3. 10/11c.
4. 10.
5. 6 ft., 7 ft., 8 ft.
6. $8(1 + \sqrt{2})$ ft.
7. $1.80.
8. 12.
9. 36 m.p.h., 30 m.p.h.
10. 40 m.p.h.
11. 3.5 in., 6.5 in.
12. 60.
13. 40 m.p.h., 150 m.p.h.
14. $2\frac{1}{4}$ sec., $2\frac{3}{4}$ sec.
15. 66c.
16. 14 m.p.h., 12 m.p.h.
17. $6(1 + \sqrt{2})$ m.p.h. = 14.5 m.p.h. app.
18. 3 P.M.
19. 23.
20. 4 m.p.h.
21. 120.
22. 40 m.p.h.
23. 4 in., 10 in.
24. 340 ft.

Exercises on Chapter X. Pages 231-236

1. (*a*) ± 2, (*b*) 0, 4, (*c*) 0, $\frac{1}{4}$, (*d*) 2, 2.
2. Add to (*a*) 9, (*b*) 121/4, (*c*) 16, (*d*) $36y^2$, (*e*) $25q^2/4$.
3. (*a*) 5/2, −7/3, (*b*) 6/5, −4/3, (*c*) 3/4, 6.
4. (*a*) 7, −3, (*b*) 8, −5, (*c*) $2 \pm \sqrt{3}$.

5. (*a*) 13, 17, (*b*) 7, -1, (*c*) 5/3, $-2/3$.
6. (*a*) 4, -11, (*b*) 9/2, -8, (*c*) 5/3, $-7/2$.
7. (*a*) 1, 3, (*b*) 7/3, -2.
8. (*a*) 10, -11, (*b*) 200, -100, (*c*) 0, 12/5.
9. (*i*) $a = 1, b = 15, c = 9$, (ii) $a = -7, b = -8, c = 17$, (iii) $a = 3, b = 10, c = 8$.
10. 6 and 7 *or* -7 and -6.
11. 7 in., 5 in.
12. 5 and 6 *or* -6 and -5.
13. 9 ft by 16 ft.
14. 0 or 10.
15. 3 ft. by 6 ft.
16. 5/6.
19. Least when $x = 2$; least value $= -4$; the line $x = 2$.
20. (*a*) 5, -1, (*b*) 6, -2, (*c*) 2, 2.
21. (*a*) $(1 \pm 2\sqrt{2})/3$, (*b*) $(2 \pm \sqrt{10})/3$ (*c*) $1/2, -5/2$.
22. (*a*) $(m \pm \sqrt{m^2 - 8n})/2$, (*b*) $(q \pm \sqrt{q^2 + 4pr})/2p$, (*c*) $(-2b \pm \sqrt{4b^2 - 6ac})/2a$.
23. (*a*) $(7 \pm \sqrt{17})a/4$, (*b*) $5a/3, -3/2$, (*c*) $(1 \pm \sqrt{5})b/2a$.
24. (*a*) ± 2, (*b*) 9 or 12, (*c*) $3 \pm \sqrt{3}$.
25. 8, -6.
26. 1, -1, 2, -2.
27. 3, -3, 5, -5.
28. 1, 3, -2, -4.
29. 16, 20.
30. 4 m.p.h.
31. 26 m.p.h., 24 m.p.h.
32. 4.
33. \$6, \$9.
34. 40 rods, 48 rods.
35. 8 in by 12 in., 6 in by 16 in.
36. 40 m.p.h., 44 m.p.h.
37. 4.7 hours.
40. $\sqrt{5} - 1$ and $3 - \sqrt{5}$; one answer.
41. 1 mile app.
43. (*a*) 2, 3, (*b*) 5.4, -0.4.
46. 36 m.p.h.
47. 12 in.
48. 12 in.
51. (*a*) for $x < 1$ and $x > 5$, (*b*) for $1 < x < 5$, (*c*) for $x = 1$ or 5 (*d*) for $x < 0$ and $x > 6$, (*e*) for $2 < x < 4$.
52. (*a*) 7, -1, (*b*) 3, 3, (*c*) 5.2, 0.8.
55. 6, $-27/17$.
56. $(-3 \pm \sqrt{3})/2$, $(-3 \pm \sqrt{17})/2$.
57. $-1, -1.5$; $0, -2.5$; $-3, 0.5$.
58. 8 in., 1 in., 9 in.
59. $1\frac{1}{2}$ ft.
61. (*a*) 0.6, -1.6, (*b*) 2.7, -0.7.

Pages 242-243

1. 33.
2. -7.
3. 0.
4. -56.
5. -3.
6. 0.

8. real and unequal.
9. no real roots.
10. real and equal.
11. real and unequal.
12. real and equal.
13. no real roots.
14. real and unequal.
15. no real roots.
16. no real roots.
17. real and unequal.
18. real and equal.
19. real and unequal.
20. $a \leqq 16$.
21. $a \geqq 4$ or $\leqq -4$.
22. $a \leqq 1$.
23. $-1 \leqq a \leqq 1$.
24. 36.
25. ± 8.
26. 9/20.
27. $\pm 5/2$.

Pages 245-246

1. $S = 5/3, P = 5/9$.
2. $S = 0, P = -11/3$.
3. $S = -2, P = -16/5$.
4. $S = 23.4, P = -32$.
5. $S = -p, P = q$.
6. $S = -m/6, P = -n/2$.
7. $x^2 + 4x - 21 = 0$.
8. $x^2 - 14ax + 48a^2 = 0$.
9. $8x^2 - 2cx - 3c^2 = 0$.
10. $x^2 - 2ax + a^2 - b^2 = 0$.
11. $x^2 - (2r + 3s)x + 6rs = 0$.
12. $x^2 - 1.8x - 4.03 = 0$.
13. $x^2 - 4x + 1 = 0$.
14. $4x^2 - 20x + 23 = 0$.
15. $x^2 - 2ax + a^2 - b = 0$.
16. No.
17. Yes.
18. (*a*) 2, (*b*) 6.
19. (*a*) 4, (*b*) 3.
20. 35.
21. $-3x$.
22. 3, 6; 18.

Pages 247-248

1. 94/25.
2. $-8/3$.
3. $-94/15$.
4. 59/4.
5. $-297/512$.
6. $-9/128$.
7. 5/3.
8. 43/9.
9. 260/27.
10. $-155/4$.
11. 33/4.
12. 145/8.

Pages 250-251

1. $x^2 - 6x - 6 = 0$.
2. $x^2 - 5x - 25 = 0$.
3. $9x^2 - 28x + 4 = 0$.
4. $5x^2 - 12x + 6 = 0$.

5. $2x^2 + 21x + 51 = 0$.
6. $x^2 + 144x + 64 = 0$.
7. $2x^2 + 9x + 10 = 0$.
8. $5x^2 - 4x - 2 = 0$.
9. $8x^2 + 184x - 125 = 0$.
10. $ax^2 - (2a - b)x + a - b + c = 0$.
11. $ax^2 + 3bx + 9c = 0$.
12. $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$.

Pages 254-255

1. $(x - \sqrt{5})(x + \sqrt{5})$.
2. $(\sqrt{2}x - \sqrt{3})(\sqrt{2}x + \sqrt{3})$.
3. $(x - 11)(x + 13)$.
4. $(3x - 8)(2x + 5)$.
5. $(4y - 9)^2$.
6. $(y - 3 - \sqrt{5})(y - 3 + \sqrt{5})$.
7. $(z + 5 - 2\sqrt{2})(z + 5 + 2\sqrt{2})$.
8. $(z - 4 - 3\sqrt{2})(z - 4 + 3\sqrt{2})$.
9. $\frac{1}{2}(2t - 3 - \sqrt{7})(2t - 3 + \sqrt{7})$.
10. $(2t + 4 - \sqrt{11})(2t + 4 + \sqrt{11})$.
11. $(5w + 6 - 2\sqrt{7})(5w + 6 + 2\sqrt{7})$.
12. $(3aw - 10b)^2$.
13. $\left(x + \frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2}\right)\left(x + \frac{p}{2} + \frac{\sqrt{p^2 - 4q}}{2}\right)$.
14. $\left(x + \frac{1 - \sqrt{5}}{2}a\right)\left(x + \frac{1 + \sqrt{5}}{2}a\right)$.

Exercises on Chapter XI. Pages 255-259

2. (*a*) real and unequal, (*b*) no real roots, (*c*) real and equal.
3. (*a*) 5, (*b*) -7, (*c*) 0.
4. (*a*) $S = 0$, $P = -5$, (*b*) $S = -6$, $P = 0$, (*c*) $S = -1$, $P = -1$, (*d*) $S = p$, $P = -q$.
5. (*a*) $S = 5/2$, $P = 3/2$, (*b*) $S = 8/3$, $P = 4/3$, (*c*) $S = -3/4$, $P = -2/3$.
6. $2x^2 - 7x + 5 = 0$.
7. (*a*) $x^2 - 3x - 54 = 0$, (*b*) $x^2 - 8ax + 15a^2 = 0$, (*c*) $x^2 - 2rx + r^2 - s^2 = 0$.

8. (*a*) real and unequal, (*b*) no real roots, (*c*) real and equal.
9. (*a*) no real roots, (*b*) real and equal, (*c*) real and unequal.
10. (*a*) 12, (*b*) 4, (*c*) 24.
11. (*a*) -10, (*b*) 5, (*c*) -1.
12. (*a*) $(x - \sqrt{3})(x + \sqrt{3})$, (*b*) $(x - 7)(x + 12)$, (*c*) $(x + 6)(x - 15)$.
15. (*a*) Wholly above x-axis, (*b*) cuts axis in two points, (*c*) cuts axis in two points, (*d*) touches axis from below.
16. (*a*) 25, (*b*) 19, (*c*) 13, (*d*) $-5/3$, (*e*) 19/9.
17. ± 8.
18. $25/24$.
19. (*a*) -13, (*b*) 22, (*c*) $1/7$, (*d*) -20.
20. (*a*) $95/2$, (*b*) 433.
21. $x^2 - 4x - 12 = 0$.
22. $2x^2 + 15x - 36 = 0$.
23. $x^2 - 12x + 24 = 0$.
24. $x^2 + 3x - 12 = 0$.
25. $x^2 - 12x + 16 = 0$.
26. $9x^2 - 118x + 81 = 0$.
27. $4x^2 - 6x - 5 = 0$.
28. $36x^2 - 52x + 1 = 0$.
29. $x^2 - 14x + 17 = 0$.
30. (*a*) $x^2 - 14x + 38 = 0$, (*b*) $x^2 - 10x + 13 = 0$, (*c*) $4x^2 - 28x + 31 = 0$.
31. (*a*) $p > 4$ or < -4, (*b*) $p = 4$ or -4, (*c*) $-4 < p < 4$.
32. (*a*) $q \leqq 1$, (*b*) $q > 1$, (*c*) $q = 1$.
33. (*a*) $(5x - 8)(2x + 15)$, (*b*) $(3x - 10)(6x + 5)$, (*c*) $(3x - 7)^2$.
34. (*a*) $(x - 2 + \sqrt{5})(x - 2 - \sqrt{5})$, (*b*) $2\left(x + \frac{5 - \sqrt{7}}{2}\right)\left(x + \frac{5 + \sqrt{7}}{2}\right)$, (*c*) $3\left(x + 2 - \frac{\sqrt{6}}{3}\right)\left(x + 2 + \frac{\sqrt{6}}{3}\right)$.
35. (*a*) $b > 6$ or < -6, (*b*) $b = 6$ or -6, (*c*) $-6 < b < 6$.
36. (*a*) ± 10, (*b*) 16, (*c*) 9/4, (*d*) ± 2, (*e*) 0 or 4.
37. $\sqrt{b^2 - 4ac}/a$.
41. (*a*) (i) upward, (ii) one point, (*b*) (i) downward, (ii) no point, (*c*) (i) downward, (ii) two points, (iii) no, (*d*) (i) upward, (ii) two points, (iii) yes.
42. 4/3.
43. $ax^2 + bkx + ck^2 = 0$.
44. $ax^2 + (b - 2ak)x + ak^2 - bk + c = 0$.

46. (*a*) $p < 0, q > 0$, (*b*) $p > 0, q + 0$, (*c*) $p < 0, q < 0$, (*d*) $p > 0, q < 0$.
47. (*a*) $b^2 - 4ac$ is the square of a rational number, (*b*) $b^2 - 4ac$ is not the square of a rational number.

Pages 262-263

1. $x = 7, y = 3$.
2. $x = 9, y = -6$.
3. $x = 1/2, y = 2/3$.
4. $x = 3/4, y = -4/5$.
5. $x = 1/4, y = 1/6$.
6. $x = 1/5, y = -1/8$.
7. $x = 1/12, y = -1/9$.
8. $x = 6, y = 8$.
9. $x = -1/14, y = 1/21$.
10. $x = 10, y = -14$.
11. $x = 3/4, y = 2/3$.
12. $x = 3, -3, 3, -3$.
    $y = 4, 4, -4, -4$.

Pages 266-268

1. $x = 3, y = 5, z = 5$.
2. $x = 6, y = 9, z = 11$.
3. $x = 8, y = 4, z = 6$.
4. $x = 14/5, y = 2, z = 11/5$.
5. $x = 2, y = 1, z = 1$.
6. $x = 4, y = 5, z = 6$.
7. $x = 7, y = 5, z = 3$.
8. $x = 9, y = -5, z = 8$.
9. $x = 3, y = 2, z = -1$.
10. $x = 8, y = 12, z = 18$.
11. $x = (a - b + c)/2, y = (a + b - c)/2, z = (-a + b + c)/2$.
12. $x = 7, y = 9, z = 11$.
13. $x = 1/2, y = 1/3, z = 1/4$.
14. $x = -2, y = 3, z = -4$.
15. $x = 30, y = 36, z = 45$.

Page 269

1. (*a*) 3, (*b*) -9.
2. (*a*) 18, (*b*) 3.
3. (*a*) 10, (*b*) 4.
4. (*a*) 9, (*b*) -12.
5. (*a*) 0, (*b*) 0.
6. (*a*) 3, (*b*) $10\frac{1}{2}$.
7. 6.
8. 8.
9. $2/3$.
10. $4/5$.
11. $9/10$.
12. $m/2$.
13. $x^2 + y^2 = 25$.
14. $x^2 + y^2 = 49/4$.
15. $x^2 + y^2 = 4k^2$.
16. $x^2 + y^2 = 9b^2/16$.

Page 273

4. Two straight lines.
5. Ellipse.
6. Hyperbola.
7. Circle.

8. Two straight lines.
9. Hyperbola.
10. Ellipse.
11. Circle.
12. Hyperbola.

Pages 278-279

1. $x = 2, 1$; $y = 1, 2$.
2. $x = 10, 1$; $y = 1, 10$.
3. $x = 9, -8$; $y = 8, -9$.
4. $x = 5, -3$; $y = 3, -5$.
5. $x = 2, 6$; $y = 3, 1$.
6. $x = 6, -6$; $y = 6, -6$.
7. $x = 1$; $y = 1$.
8. $x = 3, 4$; $y = 4, 3$.
9. $x = 1, 2$; $y = 2, 1$.
10. $x = 10, 1$; $y = 1, 10$.
11. $x = 5, -2$; $y = 2, -5$.
12. $x = 2, 0$; $y = 0, -2$.
13. $x = 4, 1.2$; $y = 3, 4.8$.
14. $x = 12, -9.2$; $y = 5, -9.2$.
15. $x = 8, 15$; $y = 15, 8$.
16. $x = -1, 2$; $y = 4, 16$.
17. $x = -2, 4$; $y = -2, -8$.
18. $x = 6, -5$; $y = 5, -6$.
19. $x = 7, 3$; $y = 3, 7$.
20. $x = 5, 7/5$; $y = 2, -26/5$.
21. $x = 2, 26/5$; $y = 6, -18/5$.
22. $x = 12, 4$; $y = 4, 12$.
23. $x = 5, -3$; $y = 12, -20$.
24. $x = 5, 15$; $y = 10, 10/3$.
25. $x = 3, -8/3$; $y = 12, -27/2$.
26. $x = 11, -115/13$; $y = 5, -107/13$.
27. 9 ft., 12 ft.

Page 282

1. $x = 6, 8$; $y = 8, 6$.
2. $x = 1, -1, 2, -2$; $y = 2, -2, -4, 4$.
3. $x = 5, -9/2, 3, -15/2$; $y = 5, -9/2, -3, 15/2$.
4. $x = 2, -2, \sqrt{2}, -\sqrt{2}$; $y = 4, -4, -3\sqrt{2}, 3\sqrt{2}$.
5. $x = 6, -6$; $y = 3, -3$.
6. $x = -6, 6, \dfrac{13\sqrt{42}}{21}, \dfrac{-13\sqrt{42}}{21}$;

$$y = 2, -2, \frac{2\sqrt{42}}{21}, \frac{-2\sqrt{42}}{21}.$$

7. $x = -5, 5, -\sqrt{3}, \sqrt{3}$
$y = 3, -3, 4\sqrt{3}/3, -4\sqrt{3}/3.$

8. $x = 9, -9$
$y = 8, -8.$

9. $x = 7, -7, 2\sqrt{2}, -2\sqrt{2}$
$y = -3, 3, 5\sqrt{2}, -5\sqrt{2}.$

10. $x = 4, -4, 2\sqrt{10}, -2\sqrt{10}$
$y = 2, -2, -2\sqrt{10}, 2\sqrt{10}.$

12. $x = 0, 9/2, 7/2$
$y = 0, 9/4, -7/4.$

13. $x = 5, -5, 3, -3$
$y = 2, -2, -4, 4.$

14. $x = 10, -10, -3\sqrt{70}/5, 3\sqrt{70}/5$
$y = 4, -4, \sqrt{70}/5, -\sqrt{70}/5.$

15. $x = 6, -6$
$y = 5, -5.$

Pages 285-286

1. 15/28.
2. 7/13.
3. 58.
4. 48.
5. 64, 46.
6. 7, 10, 15.
7. 18, 24, 30.
8. 48 years, 24 years.
9. 66 years, 41 years, 16 years.
10. \$500 at $2\frac{1}{2}\%$, \$700 at $3\frac{1}{2}\%$.
11. \$800 at 3%, \$600 at 4%, \$900 at 2%.
12. 54 lb.
13. 3 bus. bran, 5 bus. crushed oats, 2 bus. corn.
14. 5940, 4860, 3060.
15. 41 gm., 16 gm., 22 gm.
16. 11 and 9, or 15/2 and $-3/2$.
17. 15 and 7, or 2 and -6.
18. 9 and 5, 9 and -4, -8 and 5, or -8 and -4.
19. 7 cm., 24 cm.
20. 4 in by 4 in. by 10 in., or $6\frac{2}{7}$ in. by $6\frac{2}{7}$ in. by $5\frac{3}{7}$ in.

Exercises on Chapter XII. Pages 287-291

1. $x = 32, y = 15, z = -15.$
2. $x = 3, y = 1, z = 2.$
3. $x = 54, y = 76, z = 32.$
4. $x = 9, y = 11, z = 13.$
5. $x = 2, y = 3.$
6. $x = 1/2, y = -2/3.$
7. $x = 1/2, y = 1/3.$
8. $x = y = \pm\sqrt{5}/5.$
9. $x = 7, 5$
$y = 5, 7.$
10. $x = 4, 3$
$y = 3, 4.$
11. $x = 8, y = 6.$
12. $x = 12, y = 8.$

13. $x = 5, 2, -5, -2.$
$y = 2, 5, -2, -5.$

14. $x = 6, -6, 6, -6$
$y = 2, -2, -2, 2.$

15. 3/8.

16. 25/43.

17. 85, 58.

18. 258.

19. 27 years, 36 years.

20. 58 years, 30 years, 8 years.

21. 15 in., 24 in.

22. 5 in., 9 in.

23. $x = 8, y = 6, z = 4.$

24. $x = 5, y = 2, z = 1.$

25. $x = 12, y = 20, z = 30.$

26. $x = 1/2, y = 1/5, z = 1/3.$

27. $x = y = z = 0$ or $x = 12, y = 4, z = 24.$

28. $x = y = 0$ or $x = 1/7, y = 1/4.$

29. $x = 15, 77/5$
$y = 8, \frac{36}{5}.$

30. $x = 7, -2$
$y = -2, 7.$

31. $x = 2, -2, 1/\sqrt{2}, -1/\sqrt{2}$
$y = -1, 1, 3/\sqrt{2}, -3/\sqrt{2}.$

32. $x = 4, -4, 6, -6$
$y = 2, -2, 4, -4.$

33. $x = -7/4, 5/2$
$y = 5/3, -7/6.$

34. $x = 2, -2, 1/3, -1/3$
$y = -1, 1, 16/9, -16/9.$

35. $x = 4, -4, \sqrt{2}, -\sqrt{2}$
$y = 3, -3, -\sqrt{2}, \sqrt{2}.$

36. $x = 2, -2, \sqrt{2}, -\sqrt{2}$
$y = 4, -4, 3\sqrt{2}, -3\sqrt{2}.$

37. $x = 3, -3, 5, -5$
$y = 4, -4, 3, -3.$

38. $x = 2, 1, -2, -1$
$y = 1, 2, -1, -2.$

39. 8 in., 15 in.

40. Base 14 in., ht. 8 in. *or* base 8 in., ht. 14 in.

41. 12 ft. by 12 ft. by 8 ft.

42. First bought 6 lb. of each; second bought 5 lb. tea and $7\frac{1}{2}$ lb. coffee.

43. 6 of the first, 5 of the second, 8 of the third.

44. 10 gm., 4 gm., and 3 gm.

45. 2 lb. of 4% solution, 6 lb. of 12% solution.

46. 28 with B, 20 with C.

47. A does 54 pages, B, 48, and C, 91.

48. 753.

49. 3/4.

51. 15 yards.

53. 3 sec.

Page 292

1. 10^9. 2. 10^8. 3. 10^3. 4. 10^{15}. 5. 10^7.
6. 10^{12}. 7. 10^4. 8. $10^{1.77}$. 9. $10^{-0.5}$.

Pages 295-296

1. $\log 100 = 2$. 3. $\log 0.01 = -2$. 5. $\log \sqrt{10} = \frac{1}{2}$.
7. $\log s = r$. 9. $\log \dfrac{1}{10\sqrt{10}} = -\frac{3}{2}$.
11. $10000 = 10^4$. 13. $0.1 = 10^{-1}$. 15. $\sqrt[5]{1000} = 10^{3/5}$.
17. $2 = 10^{0.30103}$. 19. $h = 10^k$. 21. 1.
23. 0. 25. -6. 27. $1/5$.
29. -2. 31. -3.

Pages 298-299

1. 7.84×10^2. 3. 3.16×10^3. 5. 9.3×10^7.
7. 1.05×10^4. 9. 7.03×10^{-2}. 11. 8.4×10^{-4}.
13. 5.738×10^4. 15. 9.5×10^{-5}. 17. 6×10^{-6}.

Page 300

1. 0. 3. 2. 5. 1. 7. 0. 9. -2. 11. 4.
13. 2. 15. 1. 17. -3. 19. 0. 21. 0. 23. -5.

Page 303

1. 0.6021. 3. 1.9031. 5. 1.7404. 7. 2.8976.
9. 3.7782. 11. 2.8182. 13. 3.9365. 15. 4.8376.
17. 3.3858. 19. $\underline{3}$.8833. 21. 1.8825. 23. $\underline{2}$.7593.
25. $\underline{0}$.9317. 27. $\underline{1}$.7894. 29. 0.3030. 31. $\overline{5}$.6990.
33. $\overline{4}$.4425. 35. $\underline{2}$.9609.

Page 305

1. $\bar{2}.0185$. 2. 0.4770. 3. 0.0145. 4. $\bar{2}.5425$.
5. $\bar{1}.5821$. 6. 0.8074. 7. 2.2295. 8. $\bar{3}.7174$.
9. $\bar{2}.2736$. 10. $\bar{5}.2270$. 11. $\bar{5}.5426$. 12. $\bar{2}.3902$.
13. $\bar{1}.2931$. 14. $\bar{1}.8321$. 15. $\bar{1}.4038$. 16. $\bar{1}.7869$.
17. $\bar{1}.8673$. 18. $\bar{1}.5042$. 19. $\bar{2}.6982$. 20. $\bar{2}.7554$.

Pages 307-308

1. (*a*) 19.26, (*b*) 192.6, (*c*) 0.1926, (*d*) 0.001926.
2. 2.904.
3. (*a*) 29.04, (*b*) 2904, (*c*) 0.2904, (*d*) 0.002904.
7. 4.335. 9. 36.50. 11. 373.8. 13. 0.1523.
15. 0.004441. 17. 0.002338. 19. 0.00001799. 21. 6321.
23. 0.5131. 25. 0.01261.

Pages 309-310

1. 137.8. 3. 4627. 5. 0.02876. 7. 378.9.
9. 0.8758. 11. 3.624. 13. 1712 sq. cm.; to the nearest sq. cm.
15. 437 lb.

Page 311

1. 3.049. 3. 0.3658. 5. 0.06592. 7. 0.07015.
9. 0.07101. 11. 131.8. 13. 0.03490. 15. 7.442 in.
17. 27.47 in. 19. 175.5 ft. per min.

Page 312

1. 56.96. 3. 43.21, with doubt as to the fourth figure.
5. 0.01392, with doubt as to the fourth figure.
7. 1.003×10^{-7}.
9. 0.0830, to 3 significant figures, the fourth being in doubt.
11. 6561. 13. 796.2 cu. cm., with doubt as to the fourth figure.

Page 313

1. 92.98. 3. 3.839. 5. 0.6799. 7. 0.5394.
9. 0.3624. 11. 8.654 in.

Page 314

1. $3 \log a + 3 \log b$.
3. $\log 2 + \log \pi + \log r + \log h$.
5. $\log \pi + \frac{1}{2} \log l - \frac{1}{2} \log g$. 7. $\frac{1}{2} \log M + \frac{3}{2} \log N$.
9. $3 \log 2$. 11. $2 \log 2 + 2 \log 3$.

Pages 316-317

1. 311.0. 2. 11.35. 3. 0.5279. 4. 489.1.
5. 7117. 6. 4.717. 7. 8074, with doubt as to the fourth figure.
8. 225.5. 9. 58.41. 10. 2.408. 11. 484.2.
12. 1.177. 13. 34.34. 14. 2.009. 15. 1728 sq. cm.
16. 5501 cu. cm. 17. 783.9 cu. cm. 18. 3462 cu. units.

Exercises on Chapter XIII. Pages 318-319

8. (*a*) 251.3, (*b*) 780.5. 9. (*a*) 7.037, (*b*) 15.04.
10. (*a*) 33.31, (*b*) 74.78. 11. (*a*) 5.342, (*b*) 0.8505.
12. (*a*) 0.1889, (*b*) 0.4232.
13. (*a*) $2 \log 5$, (*b*) $4 \log 3$, (*c*) $5 \log 2$, (*d*) $\log 5 - \log 3$, (*e*) $3 \log 2 - 2 \log 3$, (*f*) $2 \log 11 - 7 \log 2$, (*g*) $\frac{1}{2} \log 5$, (*h*) $\frac{4}{3} \log 2$, (i) $3 \log 2 + \frac{1}{2} \log 3$.
14. (*a*) $\log 24$, (*b*) $\log 6$, (*c*) $\log 45$, (*d*) $\log 14$, (*e*) 1, (*f*) $\log 45$
15. (*a*) $\log b + \log h - \log 2$, (*b*) $\log 4 + \log \pi + 2 \log r$, (*c*) $\log 2 + 2 \log \pi + \log a + 2 \log b$, (*d*) $\log 4 - \log 3 + \log \pi + 3 \log r$, (*e*) $\log m + 2 \log v - \log 2$, (*f*) $\frac{1}{2}\{\log s + \log (s - a) + \log (s - b) + \log (s - c)\}$.

16. 1129. 17. 32.45. 18. 0.1928. 19. 169.5.
20. 29.99. 21. 9.27, the fourth figure being in doubt.
22. 11.73. 23. 1.345. 24. 1.497. 25. 10.32.
26. 0.7556. 27. 11.83. 28. 236.5 cu. in.
29. 1724 sq. ft. 30. 2361 cu. in. 31. 567 sq. in. 32. 5.450 in.
33. 2.065 in. 34. 1.82 sec. 35. 947.3 sq. yds.

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